The Concept of Force in Jean Le Round D'Alembert

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§ 1. Introduction

Philosophiae Naturalis Principia Mathematica, which was written by Isaac Newton in 1687, had a great influence on the 18th century mechanics. Jean le Rond d'Alembert is no exception to this rule, but as he established the three general laws, which are different from Newton's, he did not accept Newton's mechanics blindly.

As for the concept of force, which is a fundamental concept in modern day physics, d'Alembert's attitude towards Newton's concept is rather complicated. Basically, he refuses to accept Leibniz's concept of force, which was one of the two trends about this concept, and seems to accept Newton's , which consists of two categories, namely inherent force and external force. First d'Alembert enumerates la force d'inertie, which corresponds to Newton's vis insita or vis inertiae. Newton states that

> Moreover, a body exerts this force [vis inertiae] only during a change of its state, caused by another force impressed upon it, and this exercise of force is, depending on the viewpoint, both resistance and impetus: resistance insofar as the body, in order to maintain its state, strives against the impressed force, and impetus insofar as the same body, yielding only with difficulty to the force of a resisting obstacle, endeavors to change the state of that obstacle.[1]

According to d'Alembert,

> Les corps ne manifestent cette *force,* que lorsqu'on veut changer leur état ; & on lui donne alors le nom de *résistance* ou d'*action,* suivant l'aspect sous lequel on la considere. On l'appelle *résistance,* lorsqu'on veut parler de l'effort qu'un corps fait contre ce qui tend à changer son état ; & on la nomme *action,* lorsqu'on veut exprimer l'effort que le même corps fait pour changer l'état de l'obstacle qui lui résiste.[2]

We can understand from citations above that inertia was 'force' of inertia for them. Euler was the first one to declare that inertia was not force . [3]

Next, d'Alembert enumerates la cause motrice or la puissance, which corresponds to Newton's vis impressa.[4] According to d'Alembert, les causes motrices signify "les causes capables de produire ou de changer le Mouvement dans les Corps " [5]

However, we can find the point where d'Alembert's idea is different from Newton's. For example, d'Alembert dismisses the concept of force, which is a fundamental concept in Newton's mechanics. He lists three meanings of force and states the last meaning as below.

> enfin (& c'est le seul sens raisonnable) celui [sens] de l'effet même, ou de la propriété qui

se manifeste par cet effet, sans examiner ni rechercher la cause.[6]

Namely, force is measured only by effect produced by force itself. And for him, force is only "une maniere abrégée d'exprimer un fait." [7] Such a way of thinking is based on his belief that force is a ambiguous concept and mechanics must be deduced from clear ideas. Consequently, he even tries to banish this concept from mechanics.

In the next section, we will examine various meanings of d'Alembert's concept of 'force' in solving concrete problems. Also, in §.3, by examining his statements about force and his special character of concept of force, we will get some insight on his concept of force.

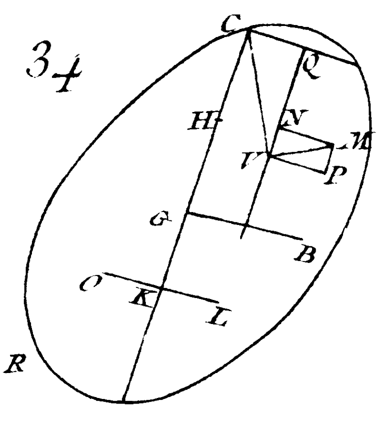
§2-1. Concept of force in rotational theory of a rigid body presented in *Traité de Dynamique.*

Let us start by examining rotational theory of a rigid body in *Traité de Dynamique.* D'Alembert presents Lemma 8 as below,

> *Soit un corps* CRM (Fig.34) *de figure quelconque, dont le centre de gravité soit* G, *& que je considerai pour plus de facilité comme une figure plane; que toutes les parties* V *de ce corps soient animées par des forces* VM *dont les directions soient perpendiculaires à la ligne* VC, *menée des points* V *à un point fixe* C *pris à volonté dans le corps, & que ces forces soient entr'elles comme les distance* VC *; je dis que la direction de la force résultante sera une ligne* KL *perpendiculaire à la droite* CG *menée par* G, *& par le point donné* C. [8]

We will briefly state his solution (Fig.1). The body rotates around a point C. First, he

Fig.1



decomposes force VM into force VN and force VP. The summation of the moment of force VN·CQ with respect to point C, which is ∫VN·CQ becomes zero. Then, he considers just the moment of force produced by VP. He names the summation of force VP 'la force résultante' and expresses 'la force résultante' by a line KL.

Next, he designates 'la force accélératrice' of the center of gravity G as φ[9]. He concludes that the summation of the moment of force in each point V with respect to the point C is equal to the moment of force when all masses concentrate on G and sets up the following equation:

$$CK\cdot\varphi\cdot MRC = \int V\cdot\varphi\cdot(VC/CG)\cdot VC$$

Here, V signifies a mass of infinitely small point V. The reason is as follows. Force F_{VP}, which acts on V along a direction VP, can be expressed as $VQ\cdot(\varphi/CG)\cdot V$. Then, 'la force accélératrice' (F_{OL}) is equal to $\int F_{VP}=(\varphi/CG)\cdot\int VQ\cdot V$. We can put $F_{OL}=(\varphi/CG)\cdot CG\cdot MRC=\varphi\cdot MRC$ because of $\int V\cdot VQ=CG\cdot MRC$, here MRC signifies a mass of the whole body. Finally, the left side of the equation is $F_{OL}\cdot CK= CK\cdot\varphi\cdot MRC$.

Since $\varphi(VC/CG)$ is equal to 'la force accélératrice' of the point V and $V\cdot\varphi(VC/CG)$ expresses force acting on the point V, the right side is equal to the summation of the moment of force in each point V.

The equation stated above means

$$N=I\cdot d\omega/dt\ldots(1)$$

,where N designates the moment of force, ω a rotational angular velocity of the body, because by defining I[10] as the moment of inertia, we can say that $I=\int V\cdot VC^2$ and $\varphi=CG\cdot d\omega/dt$.

Moreover, from the equation stated above, we can obtain the next equation,

$$CK\cdot CG\cdot MRC=\int V\cdot VC^2.$$

And we can understand that CK is the length of an equivalent simple pendulum[11] because of the equation: $CK=I/(MRK\cdot CG)$.

First, we would like to focus our attention on the fact of treating accelerated motion. Because the concept of force used here is the product of the mass of a material point, the angular velocity and the distance from a fixed point C, this concept coincides with ours. We are able to further confirm this consistency by his statement " la force suivant OL sera = $\varphi\cdot MRC$." Moreover, he names an action to rotate the body around a point C the moment of force, because he writes "le moment de la force $\varphi\cdot MRC$ agissant suivant *LKO* doit être égal (par le principe du levier) à la *somme* des momens des forces *VM* ."

Next, we will examine Corollaries 2 and 3 of this lemma. In Corollary 2, he writes as below,

> Si un corps *CRM*, entierement libre, est animé par une puissance quelconque *K*, dirigée suivant une ligne quelconque *GB*, qui passe par le centre de gravité *G*, & qu'en même tems ce corps tende à tourner autour de son centre de gravité *G*, avec une vitesse quelconque ; on prouvera, comme dans le Lemme précédent, que la force résultante sera=*K*, & dirigée (p.181) suivant une ligne *OKL* parallèle à *GB*. Donc cette puissance dirigée suivant *LKO* seroit équilibre à la puissance *K* passant par le centre *G*, & aux puissances qui tendent à faire tourner le corps. Donc le moment de cette puissance par rapport au point *G* doit être égal au moment de la puissance *K*, par rapport au même point *G*, & au moment de toutes les

puissances de rotation. Donc si on appelle Ψ la puissance qui tend à faire tourner autour *G* un point quelconque placé à la distance *b*, & *a* la somme des produits des particules par le quarré de leurs distances à *G*[12], on aura $K \times GK = K \times 0 + \Psi a/b$; ou $\Psi a/b = K.GK$[13].

We need to point out that Corollary 2 was added in the second edition (1758), and did not appeared in first edition (1743). Between the two editions, he published *Recherches sur la Précession des Equinoxes* in 1749. This publication contains Lemma 5, which is closely connected to this Corollary 2. We will discuss this point latter.

At this point, it becomes necessary to examine whether the rotation of a body treated here is a uniform or accelerated rotation. The answer is that d'Alembert applies this corollary to both kinds of rotations. Namely, 'une vitesse quelconque ' expresses a constant velocity and a variable velocity as well. In fact, he applies this corollary to a case of uniform rotation in *Recherches sur la Précession des Equinoxes*, which we will discuss in the next section. He applies it to a case of accelerated rotation in *Traité de Dynamique*, which is a problem treating a vibrating rigid body on the horizontal plane under the influence of the gravity.[14] Therefore, in the former case, 'la puissance (la force résultante *K*)' denotes momentum and in the latter case, force.

Then, in the former case, the equation$\Psi a/b = K \cdot GK$ corresponds to the equation of a rotational rigid body

$$L = I \cdot \omega \ldots (2).$$

Here, L is angular momentum. So, 'la force résultante *K*' signifies momentum and because a is moment of inertia and we can put $\Psi = b \cdot \omega$ from the equation $\Psi/b = \omega$, 'la puissance *Ψ*' signifies the velocity around the center of gravity G of the point, on which 'la puissance' to rotate the body acts. If we regard that an element of mass is omitted[15], 'la puissance *Ψ*' signifies momentum.

In the latter case, the equation$\Psi a/b = K \cdot GK$ corresponds to the equation (1). And 'la force résultante *K*' expresses force. Furthermore, because of $\Psi = b \cdot d\omega/dt$, Ψ signifies acceleration around G. In this case we can regard 'la puissance *Ψ*' as force.

In Corollary 3, he considers as follows,

De là il est aisé de conclure, pour le dire en passant, que si un corps libre & en repos est poussé par une puissance quelconque *K* suivant *OKL*, son centre de gravité *G* sera mù suivant *GB* parallèlement à *OKL*, de la même maniere que si la puissance *K* passoit par le centre *G*, & que de plus le corps tournera autour de ce même centre *G* suivant *OKL* avec une vitesse *Ψ*, telle que $\Psi . a/b = K.GK$, c'est-à-dire avec la même vitesse qu'il tourneroit, si le centre *G* étoit supposé fixe, & que la puissance *K* agît au point *K* suivant *OKL* pour faire tourner le corps.

Car (art.61) le mouvement que le corps doit prendre doit être tel, que si on le lui donnoit en sens contraire, il fût en équilibre avec la puissance *K*. D'où il s'ensuit par l'article précédent qu'il doit prendre le mouvement que nous venons de dire.

Soit *M* la masse du corps ; toutes ses parties se mouvront parallèlement à *GB* avec une vitesse égale à *K/M*, & de plus (nommant *x* leurs distances à *G*) elles tourneront suivant *OKL* autour de *G* avec une vitesse=*Ψ.x/b=K.GK.x/a* ; d'où il est aisé de voir que tous les points de la ligne *CG* auront parallèlement à *GB* une vitesse *K/M-K.GK.x/a* ; donc si on prend sur la ligne *CG* un point *H*, tel que *GH=a/M.GK*, la vitesse de ce point *H* sera nulle. Donc ce point sera en repos, & sera par conséquent ce que M. Bernoulli appelle le *centre spontané de rotation* du corps. Au reste il est visible que ce centre change à chaque instant, puisqu'à chaque instant la ligne *GK* change de situation.[16]

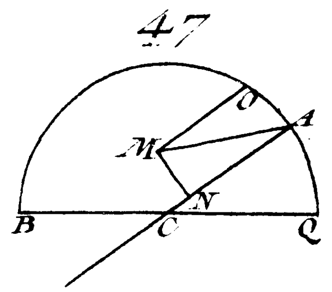
In this quote, d'Alembert employs the term 'le centre spontané de rotation' but we usually employ the term 'center of percussion'. Therefore, 'la puissance *K*' does not signify force but impulsive force or impulse. In addition, we must notice that 'la puissance *K*' acts instantaneously and the translated velocity of the center of gravity and the rotational velocity of the body around the center of gravity change discontinuously. As a matter of course, velocity *Ψ* is constant.

This corollary makes us recollect Problem 11[17] in *Traité de Dynamique*.

Supposons qu'un corps A (Fig.47) *en vienne choquer un autre* BOQ *en repos, suivant une direction* AC *qui ne passe pas par le centre de gravité* M *du corps choqué ; on demande le mouvement de ces deux corps après le choc.*[18]

According to d'Alembert (Fig.2), after collision, the body BOQ moves along the direction AC and at

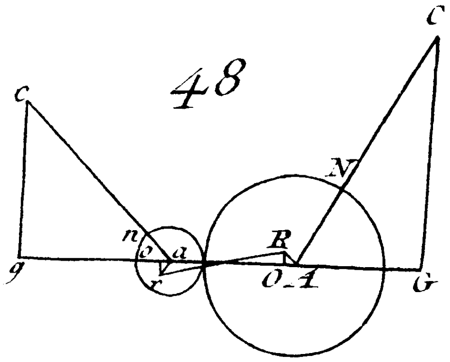
Fig.2



the same time, it rotates around the center of gravity M. The body A and the body BOQ slide each other without interaction because it is a hard body collision. He denotes the velocity of the body A before collision by *u* and its velocity after collision by u. Also, he signifies the translated velocity of the center of gravity M by α and the rotational velocity of N, which is a foot of a perpendicular dropped from M to a line AC, around M by v. Then, 1. after collision, because the body A slides on the body BOQ, we can write the equartion α+v=u. 2. If we signify the mass of the body by A, the opposite direction of the sum 'la force résultante' of the element of the body BOQ is equal to A·(*u*-u) from Lemma 8. Additionally, because it is equal to M·α, we obtain the equation M·α= A·(*u*-u). 3. If we understand moment of inertia of the body with respect to the gravity M as p, we get the rotational equation of this body, vp/MN=M·α·MN. It is clear that the situation is the same as in Corollary 3. 'La puissance K' in the Corollary 3 is equal to A·(*u*-u) and the equation Ψ·a/b=K·GK is equivalent to vp/MN=M·α·MN.

Next, he treats collision of two hard spheres in Problem 12 (Fig.3).

Fig.3



> *Deux corps sphériques* A, a, (Fig.48) *attachés aux verges* CA, Ca, *fixes en* C *&* *en* c, *se choquant avec des vitesses données, trouver leurs vitesses après le choc ; on suppose qu'avant le choc ils vont tous deux d'un même côté.*[19]

Here, he expresses velocities of the centers of sphere A and a before collision as *u* and *v* and after collision as u and v, respectively. Since this collision is a hard body collision, components of u and v along the direction gG are equal and we have the first equation u·CG/CA=v·cg/ca.

Next, he defines F and f, which correspond to the moment of inertia of the sphere A and a with respect to C and c, respectively. Then, from d'Alembert's principle, we obtain the second equation

F· (*u*-u)/AC·CG+f(*v*-v)/ac·cg=0

Bezot annotates this equation as follows.[20] Since the quantity *u*-u is the velocity lost by the center of the sphere A, (*u*-u)·CM/CA is the velocity lost by any point M on this sphere. Then, (*u*-u)·CM/CA·M expresses the force lost by M, where M signifies the mass of the point M. It is clear that he names a change of momentum 'la force'. He replaces the sum of forces at any point by one force, which acts on the point G along to the direction GA. By putting this force by X, we obtain the equation

∫(*u*-u)·CM/CA·M·CM=X·CG…(3)

From (3), we have the equation X= F· (*u*-u)/AC·CG. We can obtain the second term f(*v*-v)/ac·cg similarly. When we express the distance of an element of the rigid body from the rotational axis as r, the left side of equation (3) signifies ∫m·du·r. In Corollary 2 of this problem, he writes the following:

> Mais dans Sphère qui tourne autour d'un point fixe, la quantité de mouvement & la force ne sont pas la même chose : il faut avoir égard de plus au bras de levier par lequel chaque particule agit ; c'est la somme des produits de chaque élément par sa vitesse & par sa distance au point fixe, qui fait la force, & non pas seulement la somme des produits de

chaque élément par sa vitesse.[21]

Therefore, equation (3) expresses the force lost by the rigid body A.

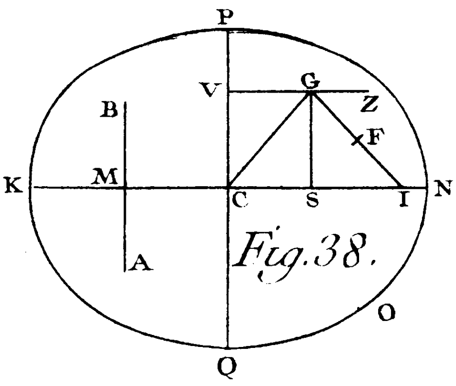
§ 2-2. Concept of force in the second resolution of *Recherches sur la Précession des Equinoxes*

In §3, we examine d'Alembert's concept of force, which differs depending on uniform rotation, accelerated rotation (continuous change in rotational velocity) and discontinuous change in rotational velocity of a rigid body. In this section, we will show that d'Alembert uses the same concept of force for uniform rotation and accelerated rotation of a rigid body when he treats the precession of the earth in chapter 11 of *Recherches sur la Précession des Equinoxes*. We will start with Lemma 5.

> *Soit un corps* PKO (Figure 38) *dont le centre de gravité soit* C, CQ *une ligne tirée à volonté par le centre* C, KCN *un plan perpendiculaire à* CQ*; & soit menée dans ce plan une ligne perpendiculaire à* KN, *& que l'on prendra pour l'Axe du corps; desorte que* PKO *représente la section de ce corps par un plan perpendiculaire à l'Axe. Je dis que si tous les points* G *du solide sont animés par une même force ¢ qui agisse suivant une direction parallele à* CQ, *& outré cela par des forces* GF, *dont la direction soit perpendiculaire aux distances des points* G *à l'Axe, & parallele au plan* PKO, *& qui soient proportionnelles à ces distances; la direction de la force unique qui résulte de toutes celles-là, sera parallele à CQ, & que cette force sera* $=\int G \times ¢$, *c'est-à-dire sera la même que si les forces suivant* GF *étoient nulles.*[22]

The figure PKQN (Fig.4) is a cross-section perpendicular to the rotational axis of a rotating

Fig.4



body. According to d'Alembert, two different types of forces act on this body. One is an attractive force acting along the direction CQ, which is parallel to this plane. He expresses this force, acting on any mass element, as ¢. The other is called 'la force de rotation', which acts on any mass element G. This force has a direction GF, perpendicular to CG, and a magnitude that is proportional to the distance CG. It corresponds to 'la force VM' in §2.

According to this Lemma, when we sum up these two types of forces acting on this body, the summation of 'la force de rotation' over the whole body is equal to zero and we can only obtain $\int ¢\cdot G$ as a result. Next, in Corollary 1, he proves that the resultant force, which comes from the summation of these forces over the whole body, exists on the equatorial plane of this body. In Corollary 2, he expresses this resultant force by BMA (see Fig.4) and names it F. In 'Remarque', he demonstrates that this force F does not pass through the center of gravity C, and obtains the equation.

$$F\cdot CM=\int \pi\cdot G\cdot CG^2$$

Here, π designates 'la force de rotation' at a unit distance from C and G designates an element mass.

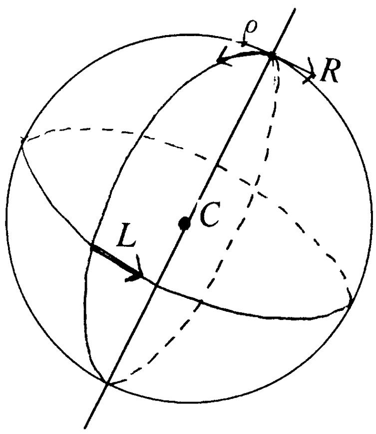
This equation is derived from the next equation,

$$F\cdot CM=\int G\cdot\varphi\cdot CS+\int \pi\cdot G\cdot CG^2$$

by setting that the first term of the right side equals to zero. This equation means that the sum of the moment of 'la force de rotation' and the attractive force φ around C is equal F·CM.[23]

Next, he conceives of a force L[24] (Fig.5), which acts on the extremity of the sphere's

Fig.5



equatorial plane, so as to establish the equation L× (a radius of the sphere)=F·CM. Namely, he replaces moment F·CM by another moment of force L. So far, a precession does not exist.

He introduces a precession as follows. In actuality, because the earth is a spheroid, its pole experiences two forces, which are indicated by R and ρ in the figure, during an infinitesimal time dt. During an infinitesimal time dt, the equatorial plane changes its situation, passes through the center of gravity C, and contains the resultant force of L, R and ρ. Here, d'Alembert supposes that L is much larger than R and ρ. Then he combines L and R, so that the axis of the earth changes to a direction perpendicular to R. Next, he combines L+R and ρ, so that the axis changes to a direction perpendicular to ρ. At this point, the precession of the earth takes place.

And in the final part of this chapter, he discusses as below,

> Enfin, si on veut avoir l'équation de la vitesse de rotation, on remarquera que la force *L* devient dans l'instant suivant *(R+L)×(1-R/L)=L-RR/L* (art.98) : or *RR′* étant censée infiniment petite du second ordre, il s'ensuit que la force de rotation ne change pas

sensiblement, & que par conséquent on peut regarder la vitesse de rotation comme constante ; ce qui confirme ce que nous avons remarqué dans l'article 77, que la vitesse de rotation de la terre n'est point sensiblement altérée par l'action de la Lune & du Soleil.[25]

From this statement, we can conclude that d'Alembert applies this lemma to both uniform rotation and accelerated rotations. Therefore, we must interpret the equation $F\cdot CM=\int\pi\cdot G\cdot CG^2$ in two ways stated above.

Because π does not relate with integration, we can put $\int\pi\cdot G\cdot CG^2=\pi\int G\cdot CG^2=\pi\cdot I$ (I: moment of inertia). Then, we obtain the equation

$$F\cdot CM=\pi\cdot I\ldots(4)$$

or $L\times$ (a radius of the sphere)$=\pi\cdot I$.

In the case of the earth, which rotates uniformly, neither 'la force de rotation', nor the velocity of rotation changes. Therefore, we can write the equation $\pi=1\cdot\omega$, because π expresses the angular velocity of a point whose distance is 1 from the center of rotation. Finally π expresses a velocity of the point. So, equation (4) corresponds to

$$L=I\cdot\omega\ldots(2)$$

Next, in the case of accelerated rotation, π expresses $1\cdot d\omega/dt$, namely angular acceleration of the point and F expresses force. Therefore, equation (4) corresponds to

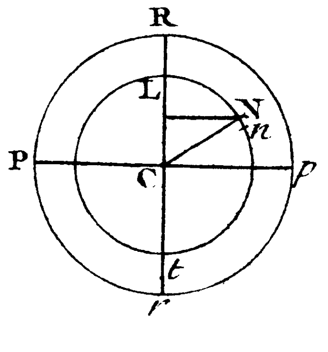
$$N=I\cdot d\omega/dt\ldots(1)$$

Let us now examine another example in equation (4), which shows a confused treatment between uniform and accelerated rotations of a rigid body. D'Alembert discusses Problem 9 in the chapter 11of *Recherches sur la Précession des Equinoxes* as follws :

On suppose que tous les points d'une Sphere PRpr (Fig. 60) *soient animés par des forces qui tendent à la faire tourner autour de l'Axe* Rr, *& qui soient par conséquent proportionnels à leurs distances à cet Axe, & que* Ψ *soit la force qui agit à la distance 1 du centre* C ; *on demande la somme des produits de chaque particule par sa force accélératrice & par sa distance à l'Axe* Rr.[26]

To solve this problem, he imagines a circle LN (Fig.6) parallel to an equatorial plane Pp and

Fig.6



expresses ‘chaque particule’ as an element of a circle whose radius is LN, ‘sa force accélératrice’ as Ψ·LN/CP, ‘sa distance à l'Axe Rr’ as LN. First, he integrates the sum of products with respect to a disk that passes through L and is parallel to Pp. He then, by integrating it from R to r, obtains $8\pi a^5/15$ (a being the radius of the sphere). Supposing that the density is 1, and because $I=8\pi a^5/15$, we obtain the following equation:

$$8\pi\Psi a^5/15=I\cdot\Psi$$

He continues in the corollary of this problem as follows. “Donc si Π est une force appliquee à l'extrémité *p* du rayon *Cp* & perpendiculaire au plan *RPp*, on aura (*art.* 90) *Ψ×K×4D=Π×1*, & *Ψ=Π/K.D.*”[27]

Therefore, we obtain equation (5).

$$\Psi\times K\times 4D=\Pi\times 1\ldots(5)$$

Here, K×4D corresponds to $8\pi a^5/15(=I)$, if we were to suppose that the sphere consists of concentric spherical layers. In addition, he supposes that the radius of the sphere is 1.

Because force Ψ causes ‘la force accélératrice’, by neglecting the mass, we obtain the next equation

$$\Psi=dv/dt=d\omega/dt$$

, where v is the velocity of an element at a distance 1 from the axis. So, we can rewrite equation (5) as Ψ×I=Π×1, and this expression is equivalent to

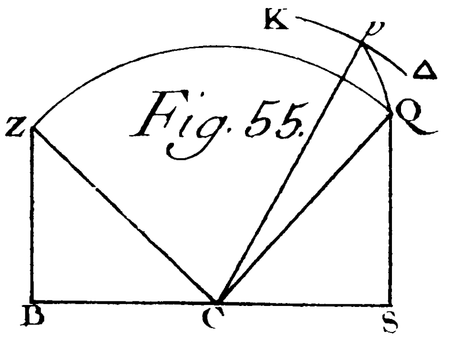
$$N=I\cdot d\omega/dt\ldots(1)$$

But, he applies the result of this corollary to the next problem, which deals with the uniform rotation of the earth.

> *Déterminer le changement que l'action du Soleil & celle de la Lune doivent produire dans la position de l'Axe* de rotation *de la terre, & dans la rotation de la terre autour de ce même Axe.*[28]

In the figure 7, the sphere is rotating around the earth's axis CQ at any instance dt. BS is contained in

Fig.7



the ecliptic plane and a plane QBS is perpendicular to the ecliptic plane. At this point, d'Alembert comments:

Or faisons (*art*.43) $dt=u\cdot dz/g$, & soit $k.g\times 1/u$=à la vitesse de rotation d'un point quelconque de l'Equateur terrestre ; si on réduit toutes les forces de rotation de toutes les particules du globe, à une seule qui passe par le point *L* & qui soit perpendiculaire au cercle *CLQ*, cette force j'ai ci-dessus appellée *L*, sera (*art*. 90& 115) $kg/u\times 4D\times K$.[29]

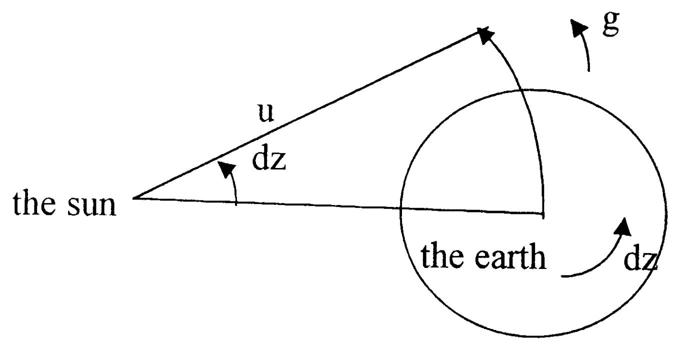
According to art.43,

...dont l'intégrale est $dP=-d\varepsilon\sqrt{[1-yy]}+$ une costante infiniment petite *dZ*.

Cette constate *dZ* est évidemment proportionnelle au tems *dt* qui est supposé constant ; donc si on suppose que la terre se meuve d'un mouvement uniforme autour du Soleil, & que durant le tems *dt* elle parcoure l'angle *dz*, on pourra supposer $dZ=kdz$, *k* exprimant une constante indéterminée. ... on remarquera 1°. que si on appelle *g* la vitesse de la Terre autour du Soleil à la distance *u* que l'on regarde comme constante, on aura $dt=u\cdot dz/g$; ...[30]

Here (Fig.8), he supposes that during an infinitesimal time dt, the earth revolves around the sun an

Fig.8



infinitesimal angle dz. At the same time, the earth rotates an infinitesimal angle dZ. Moreover, he sets the revolving velocity as g, and the distance between the earth and the sun as u. Then, we can obtain the next equation.

$$g\cdot dt=u\cdot dz$$

Substituting $dZ=k\cdot dz$, we obtain the next equation.

$$k\cdot g/u=dZ/dt$$

Because he supposes that the radius of the earth is equal to 1, 'la vitesse de rotation d'un point quelconque de l'Equateur terrestre' ($k\times g\times 1/u$) is equal to dZ/dt. Furthermore, we can set up the equation $k\times g/u=\omega$ (ω: angular velocity, which is constant), because the earth rotates uniformly. Then, the equation

$$L= kg/u\times 4D\times K\ldots(6)$$

corresponds to

$$L=I\cdot\omega\ldots(2)$$

if we were to interpret the left side of equation (6) as $L\cdot 1$.

D'Alembert describes in the Corollary4 of the Lemma 5,

> ...Donc, si la force *F* agit suivant *BA*, elle produira deux mouvemens, l'un suivant *CQ* parallele à *BA*, tel que la force ¢ qui anime toutes les parties *G*, soit *F*/∫*G*, l'autre de rotation autour d'un Axe perpendiculaire au plan *BMC* ou *MCQ*, & qui sera tel que la force de rotation π à la distance 1, sera = $F.MC/\int G.CG^2$.[31]

Here, by putting $\int G.CG^2=I$, we obtain equation (4).

$$F \cdot CM = \pi \cdot I \ldots (4)$$

Therefore, the general rotational equation of a sphere (4), which is produced in Corollary 2, expresses both uniform rotation and accelerated rotations. This formula is equivalent to (5) in case of accelerated rotation and equivalent to (6) in case of uniform rotation, respectively. Also, π corresponds to Ψ in the former case and to $k \cdot g/u$ (a constant rotational velocity of an equatorial point) in the latter case, respectively. Similarly, F corresponds to Π and L, respectively. Namely, π expresses 'la force accélératrice' in the former case and the velocity in the latter case, F expresses force and momentum, respectively.

Now, let us summarize our discussion (Table 1). In the table, the term 'Δ' signifies a finite

Table 1. D'Alembert's concept of force

	Uniform motion (rotation)	Variable motion (rotation)	
		discontinuous	Continuous
Material point	$m \cdot u$ (momentum)	$m \cdot \Delta u$ (impulse)	$m \cdot \varphi$ (force)
Rigid body	$\sum mu \cdot r (= I \cdot \omega)$ (angular momentum)	$\sum m \cdot \Delta u \cdot r (= I \cdot \Delta\omega)$	$\sum m \cdot \varphi \cdot r (= I \cdot d\omega/dt)$ (moment of force*)

Modern terms are indicated in ( ).

* In this case, d'Alembert names this quantity 'le moment de la force', not 'la force'.

change and 'd' signifies an infinitesimal change. In §2-1, by solving Lemma 8 of *Traité de Dynamique,* d'Alembert treats accelerated rotation of a rigid body and 'la force' of each element of the body, for example 'la force VM', as equivalent to the modern notion of force. And in Corollary 2, he expresses accelerated rotation and uniform rotation with the same equation. In the former case, 'la force' is equivalent to force and in the latter it is equivalent to momentum. Moreover, in Corollary 3, he discusses a discontinuous change in rotation, in which case, 'la puissance' signifies impulse. In Problem 11 of *Traité de Dynamique,* which is related to this corollary, 'la force résultant' resulting from the sum of the force of each element is equivalent to the change in momentum and in the end equivalent to impulse. The situation is the same in Problem 12 and 'la force perdue' signifies momentum lost.

In §2-2, in discussing Lemma 5 of *Recherches sur la Précession des Equinoxes*, he gets an equation, which expresses both accelerated and uniform rotation. He then applies this equation to the uniformly rotating earth. As a result, his concept of force is quite different from ours and the uniformly rotating earth has 'la force L', which is equivalent to momentum, as well.

We can conclude from his discussion about a material point that in the case of variable motion, he has different formulas of force for discontinuous and continuous change in motion. In the former case, 'la force' signifies change in momentum ($m\Delta u$), which can be seen in Corollary 3 (art. 142), Problem 11 and Problem 12 in *Traité de Dynamique*. In the latter case, it signifies force ($m\varphi=mdu/dt$), which can be seen in Lemma 8. Finally, for him, force signifies a potential ability to change motion of a body or affect obstacles, so a uniformly moving material point has force too, which is equivalent of momentum.[32]

In case of a rigid body, we must consider the note 21. For him, a sphere rotating around a fixed point has 'la force', which signifies the sum of the product of the mass, the velocity and the distance from a fixed point of each element. Therefore, in this case, it signifies angular momentum, which can be confirmed by Problem 10 in *Recherches sur la Précession des Equinoxes*. Next, in case of discontinuously accelerated rotation, it signifies $\sum m\cdot\Delta u\cdot r=\sum m\cdot r\cdot\Delta\omega\cdot r=I\cdot\Delta\omega$. We can confirm these relations in Corollary 3 (art. 142), Problem 11 and Problem 12 in *Traité de Dynamique*. Finally, in case of continues change of rotation, it is equivalent to $\sum r\cdot m\phi=\sum r\cdot m\cdot du/dt=\sum r^2\cdot m\cdot d\omega/dt=I\cdot d\omega/dt$, which can be confirmed by Lemma 8 in *Traité de Dynamique* and Problem 9 in *Recherches sur la Précession des Equinoxes*. In this case, d'Alembert does not name it 'la force', but 'le moment de la force'.

§.3. D'Alembert's theoretical consideration of force.

By considering that he regards 'la puissance' in the same light as force[33], we can realize how he interprets force as below.

> On ne doit donc entendre par l'action des *puissances,* & même par le terme de *puissance* dont on ne sert communément en Méchanique, que le produit d'un corps par sa vîtesse ou par sa force accélératrice.[34]

Then, what is force for d'Alembert, $m\cdot u$ or $m\cdot\Delta u$ or $m\cdot\varphi$ (φ: la force)?

Hankins classifies this confused concept of force into two groups according to two kinds of causes that induce change in the motion of a body. Namely, one induces discontinuous change of motion like a percussion and the other induces continuous change of motion like accelerated motion. In addition, he interprets change of momentum as force in the former case, and the product of the mass and acceleration as force in the latter case. Therefore, in the latter case, the definition of force coincides with ours. Hankins tries to ascribe the cause of this classification to d'Alembert's concept of body. D'Alembert theoretically presupposes the existence of hard bodies. Then, when ideal bodies collide with each other, their velocities change instantaneously, so in this case, if we measure force by the rate of change of momentum, force would become infinite. Therefore, it is impossible to express change of motion with one equation. Hankins describes Newton's second law as below,

> Since the second law is a law about impressed forces, it has to cover *all three kinds* of impressed force: percussion, which causes an instantaneous change in momentum; pressure, which is applied continually and produces an acceleration; and centripetal force, or motion caused by an attracting centre. Pressure and attraction cause acceleration, while percussion produces a single discrete change in momentum. Newton was caught in the quandary described at the beginning of this chapter. He had to choose between forces that were proportional to discrete finite changes of momentum and forces that were proportional to accelerations. The former were produced by the impact of hard bodies and the latter were characteristic of falling bodies and planetary motion. Where d'Alembert divided the problems of mechanics into two different categories, Newton attempt to subsume them all under the same set of laws.[35]

Let us now examine d'Alembert's statements to judge whether Hankins' consideration is right. According to d'Alembert, all changes in motion can be divided into two kinds, namely changes that are either explicable or inexplicable by impulse.

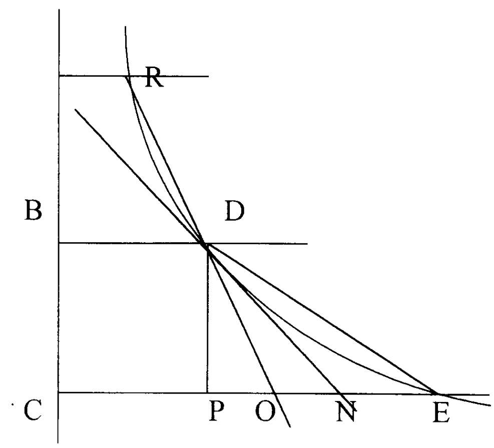
> Nous ne connoissons que deux sortes de causes capables de produire ou d'altérer le mouvement dans les corps ; les unes viennent de l'action mutuelle que les corps exercent les uns sur les autres, à raison de leur impénétrabilité : telles sont l'impulsion & les actions qui s'en dérivent, comme la traction...
>
> On peut donc regarder l'impénétrabilité des corps comme une des causes principales des effets que nous observons dans la nature ; mais il est d'autres effets dont nous ne voyons pas aussi clairement que l'impénétrabilité soit la cause ; parce que nous ne pouvons démontrer par quelle impulsion méchanique ces effets sont produits ; & que toutes les explications qu'on en a données par l'impulsion, sont contraires aux lois de la méchanique, ou démenties par les phénomenes. Tels sont la pesanteur des corps, la force qui retient les planetes dans leurs orbites, &c [36]

Here, change in motion by impulse can be divided into two kinds, namely, discontinuous finite change in velocity like percussion and continuous change in velocity like accelerated or restarted motion.[37] According to d'Alembert, we can perceive a cause for change in motion in the former ("...il n'y a tout au plus que l'impulsion seule dont nous soyons en état de déterminer l'effet par la seule connoissance de la cause... "[38]). Then, by supposing the existence of hard bodies, he gets the velocities of bodies after collision by using the law of balance. In this case, we can determine the motion of the bodies deductively.[39]

In contrast, in the latter case, we can only perceive the effect as changed motion, because the cause is unknown. In this case, he obtains the equation $du=\pm\varphi \cdot dt$, here the "+" signifies accelerated motion and "–" retarded motion, respectively. He further divides this case into two sub-cases. The one is that motion changes according to an arbitrary and hypothetical law and the other is that motion can be discovered by an experiment, like in the case of a falling body.

Let us examine how d'Alembert gets the equation $du=\pm\varphi \cdot dt$ in the former sub-case. He discusses this matter in art. 15-21 of *Traité de Dynamique* as follows. When any body moves on a curved line, he considers the displacement of the body during an infinitesimal time dt and approximates this small displacement in two ways. The one is that during dt, the body changes its velocity continuously and describes an arc of a circle. Therefore, the body describes a succession of arcs of circle, which are connected smoothly at joints. He names this curve 'la courbe rigoureuse'. The other is that the body changes its velocity discontinuously and instantaneously at the beginning of each dt Therefore, the body describes a polygon with infinitesimal sides. He names this polygon 'la courbe polygone'. In Fig.9, the ordinate expresses time and the abscissa expresses displacement

Fig.9



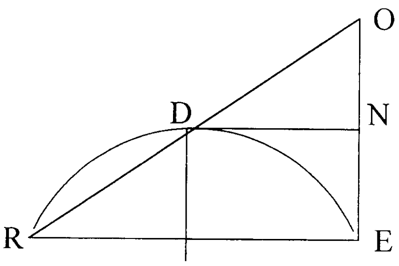
of the body. By using the courbe rigoureuse, the body describes arcs RD and DE and by using the courbe polygone, it describes chords RD and DE. We put the infinitesimal time BC as dt, the foot dropped from point D to CE as P, the crossing between the extension of chord RD and CE as O and the crossing between a tangent to arc RD or DE at the points D and CE as N. In the courbe polygone approximation, the body receives an impact at each apex and proceeds uniformly to a next apex on the chord. Consequently, the body receives an impact at the beginning of an infinitesimal time BC (dt) at point D instantaneously and proceeds on chord DE uniformly.

First, in the courbe rigoureuse approximation, using an elemental geometry, he demonstrates that a value of NE/BC^2 is constant during an infinitesimal time and puts it as F.[40] Here, NE signifies the displacement of the body during an infinitesimal time BC by acceleration. That NE is proportional to BC^2 signifies that the body has a uniformly accelerated motion during an infinitesimal time.

In the courbe polygone, the displacement of the body by an impulse at point D is OE. And he demonstrates a relation OE=2NE in these two approximations. Therefore, by putting the distance BD as e, he obtains $dde/dt^2=2F$, because of dde(=OE)=2NE and $NE/BC^2=F$. And by putting $2F=\varphi$, he obtains $\varphi dt^2=\pm dde$. By using u=de/dt and regarding dt as constant, he finally gives $\varphi dt^2=\pm du$.

Next, we will examine his approximation for the curbed motion of a body. First, we must point that he uses the courbe polygone to obtain$\varphi dt^2=\pm du$. These two approximations are reminiscent ofl E.J. Aiton's discussion about a dispute between G.W. Leibniz and P. Varignon.[41] He discussed the two ways of approximations of any curbed motion of a body, namely arc of a circle and polygon approximations. According to his discussion, at first, both Varignon and Leibniz used polygon approximation and the former regarded the motion of a body on chord DE as composed of a uniform motion DO and a uniformly accelerated motion OE (Fig.10). The latter regarded this motion as

Fig.10



composed of a uniform motion DN and a uniform motion NE. But Leibniz accepted Varignon's remark and regarded this motion as composed of a uniform motion DO and a uniform motion OE. Finally, Varignon pointed that composition of a uniform motion DN and a uniformly accelerated motion NE gives the real motion of a body on an arc of circle DE. But Leibniz never accepted his suggestion. Varignon discussed that polygon approximation did not express the real motion of a body correctly but it was equivalent to arc approximation as far as both approximations gave the same result. From this point of view, even polygon approximation gives a correct result only when we used coherently. Johann Bernoulli gave the same discussion in his letter.[42]

D'Alembert adopts Leibniz's approximation, by considering an instantaneous impulse to a body at each apex. But he admits that in arc approximation the body continues to receive impulses while describing the arc DE and states as below.

> De là il s'ensuit, que puisque dans la courbe polygone l'effet de la puissance accélératrice est représenté par un mouvement uniforme, on ne doit point supposer dans cette hypothese que la vitesse du corps s'accélere par degrés pendant l'instant *BC*, mais qu'au commencement de cet instant *BC*, lorsque le corps a parcouru l'espace *BD*, sa

vitesse reçoive brusquement & comme d'un seul coup toute l'augmentation ou la diminution qu'elle ne doit réellement avoir qu'à la fin de l'instant *BC*.[43]

Furthermore,

...dans ce cas [la courbe rigoureuse], la vitesse est censée s'accélérer ou se retarder uniformément pendant tout le cours de l'instant BC[dt], en vertu de la puissance accélératrice, qui est censée donner au mobile pendant cet instant une suite de petits coups égaux & réitérés; & la somme de ces petits coups est égale au coup unique, que la même puissance est censée donner au corps dés le commencement de l'instant BC dans l'hypothese de la courbe polygone.[44]

He gives OE=2NE, because in the courbe polygone approximation, the body obtains a velocity at the beginning of an infinitesimal time dt and proceeds with uniform motion, while in the courbe rigoureuse, the body obtains at the end of dt, and proceeds in a uniformly accelerated motion. Although d'Alembert uses Leibniz's approximation, he sates that both approximations give the same result if we use them coherently, just like Johann Bernoulli.

Si un des effets est calculé dans l'hypothese de la courbe rigoureuse, il faut calculer l'autre dans la même hypothes ; autrement on courroit risque de faire le rapport des forces, c'est-à-dire de leurs effets, double de ce qu'il est réellement.[45]

Next, he searches an equation of accelerated motion discovered by experiment from an experimental law of the fall of a body and gives $e=at^2/T^2$ for expressing a curve ADE. Here a expresses the space moved by the body during an arbitrary constant time T. After differentiating $e=a\cdot t^2/T^2$, we obtain equations $dde=2a\cdot dt^2/T^2$ and $du=2a\cdot dt/T^2$.

As a conclusion, he declares as follows.

Il est donc évident que quand la cause est inconnu, l'équation $du=\pm\varphi dt$, est toujours donnée.[46]

Therefore, he states in *l'Encyclopédie* that

M. Leibnitz est le premier qui se soit servi de ce terme [la dynamique] pour désigner la partie la plus transcendante de la méchanique, qui traite du mouvement des corps, en tant qu'il est causé par des forces motrices actuellement & continuellement agissantes. Le principe général de la Dynamique prise dans ce sens, est que le produit de la force accélératrice ou retardatrice par le tems est égal à l'élément de la vîtesse ; la raison qu'on en donne est que la vîtesse croît ou décroît à chaque instant, en vertu de la somme des petits coups réitérés que la force motrice donne au corps pendant cet instant.[47]

But, because $\varphi dt^2=\pm du$ is given hypothetically or experimentally, it is less certain than the law of percussion. He opposes regarding this equation as a principle and regards it as a definition of 'la force accélératrice.' He even states as below.

> J'espere qu'on verra dans la seconde Partie de cet Ouvrage, que non-seulement ce prétendu principe [$\varphi dt^2=\pm du$] est encore inutile dans ce cas [les causes sont connues], mais que l'application en est insuffisante & pourroit même être fautive.[48]

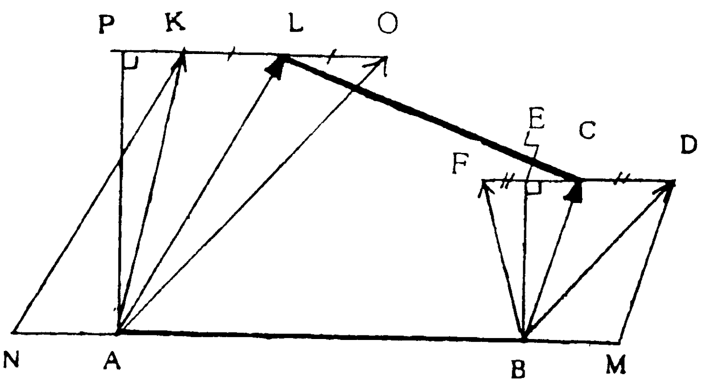
Only the law of percussion is suitable for d'Alembert's deductive mechanics.

> Le mot Dynamique est fort en usage depuis quelques années parmi les Géometres, pour signifier en particulier la science du mouvement des corps qui agissent les uns sur les autres, de quelque maniere que ce puisse être, soit en se poussant, soit en se tirant par le moyen de quelque corps interposé entr'eux, & auquel ils sont attachés, comme un fil, un levier inflexible, un plan, &c.
>
> Suivant cette définition, les problèmes où l'on détermine les lois de la percussion des corps, sont des problèmes de Dynamique.[49]

In fact, he solves some problems by regarding them as a kind of collision in *Traité de Dynamique,* Second Part. For example, in Problem 10, when discussing a case where a material point slides down an inclined plane, he regards an interaction between the inclined plane and the material point as a kind of collision.[50]

We will give another example in *Traité de Dynamique*, where he regards the motion of a body as a kind of collision. At the beginning of the Chapter 4 of the Second Part, he discusses the motion of a weightless bar connecting material points at the end of it and searches a condition for the conservation of vis viva. Craig G. Fraser discusses this process in detail[51]. We will therefore examine it briefly. Two material points A and B are connected by a weightless bar AB (Fig.11). He supposes

Fig.11



that infinitesimal velocities AK and BD are given to the material points A and B, respectively. But since the two material points are restricted, he supposes that in reality the material points A and B have velocities AL and BC, respectively. And velocities KL and DC are lost velocities because of restriction. By d'Alembert's principle, the lost motions of the system are in equilibrium. Therefore, he obtains an equation,

$$B \cdot BC^2 + A \cdot AL^2 = A \cdot AK^2 + B \cdot BD^2 - A \cdot KL^2 - B \cdot CD^2,$$

where A and B signify masses of bodies A and B, respectively. Further in Corollary 1 he states that when we can ignore lines KL and CD, namely lost velocities, we obtain the relation $B \cdot BC^2 + A \cdot AL^2 = A \cdot AK^2 + B \cdot BD^2$ and vis viva is conserved. Additionally, in Corollary2, where we cannot ignore lines NA(=KL) and BM(=CD), by deciding points F and O, which satisfy relations CF=CD and LO=LK, and using the Pythagorean theorem, he gets the relation

$$B \cdot BF^2 + A \cdot AO^2 = B \cdot BD^2 + A \cdot AK^2$$

and states that in this case, vis viva is conserved as well. At the end of the Corollary2, he adds that "mais, si l'on y fait attention, ce cas est précisément celui du choc de deux corps élastiques (art.168)."[52]

By considering this addition, we will examine the Corollary1. First, he writes as below in art.168.

> Donc si tant de corps durs qu'on voudra se choquent à la fois, & que *a, b* &c. soient leurs vitesse avant le choc, qui soient changées après le choc en a, b &c. alors regardant les vitesse *a, b* &c. comme composées des vitesses a, α ; b, ζ ; &c. les vitesses de ces mêmes corps après le choc (dans le cas où ils seront élastiques) seront composées des vitesses a, -α ; b, -ζ ; &c.[53]

By applying art.168 to the Corollary 1, we can understand that *a*, a and α correspond to AK, AL and KL, respectively. Therefore, if AO represents the velocity after an elastic collision, AL will represent the velocity after a hard body collision. The situation is the same for body B. Consequently, Corollary 1 signifies not only that vis viva is conserved when lost velocities can be neglected, but also that the motion of the system can be regarded as a succession of hard body collisions. Furthermore, he demonstrates geometrically in art.190 that except for the beginning of movement, velocities of the bodies change only infinitesimally during dt as long as no external force acts on the system. In this way, he thinks that when causes are known, continuous motion can be explained by hard body collision.

§.4. Conclusion.

According to Cohen, the force stated by Newton in his second law signifies impulse but Newton realized that continuous force could be expressed by this law. And the reason why Newton regarded impulsion as fundamental was to make his theory more easily accepted by his contemporaries.[54] Furthermore, in the 17^{th}-18^{th} centuries, usually dt was regarded as constant, a concept of force could be expressed variously. Cohen also gives two expressions for Newton's second law, namely F%dV and F%dV/dt[55], and states that Newton chose these expressions according to problems.

The reason why d'Alembert considers impulsion as fundamental is not the same as Newton's. As stated above, for d'Alembert, only impulsion can be a known case of change of motion. And when the velocity of a body changes by one impulsion, if we measure force by $m\varphi=mdu/dt$, force becomes infinite. For du is a finite quantity, but dt is an infinitesimal small quantity. Therefore it is reasonable to measure force by $m\Delta u$. When the velocity of a body changes continuously, he postulates that at the beginning of an each infinitesimal time dt, the body receives a small impulsion and obtains the relation $m\varphi=mdu/dt$. In this case, change of velocity is the first order infinitesimal as dt Therefore, since force $m\varphi=mdu/dt$ has a finite value, this measure of force is reasonable. Furthermore, such a postulate does not contradict his belief in the existence of hard body, because in hard body collision, the velocity changes instantaneously (even from the first order infinitesimal). Then, we can accept Hankins' explanation.

According to Cohen, Newton derived different meanings for a concept of force from the fact that dt was regarded as constant and an equal sign meant proportionality at those days. This is the case for d'Alembert. For example, although he states that force is equal to $m\varphi=mdu/dt$ in note 34, he writes as below.

> Ainsi nous entendrons en général par la force motrice le produit de la masse qui se meut par l'élément de sa vitesse, ou, ce qui est la même chose, par le petit espace qu'elle parcourroit dans un instant donné en vertu de la cause qui accélere ou retarde son Muvement ;[56]

Namely, la force is equal to $mdu=m\varphi dt$ or $mdde=m\varphi dt^2$. Such a variety in the meaning of force is understandable by considering the special circumstances in those days, which Cohen discusses.

However, the situation is rather complicated for d'Alembert. He tries to abandon the concept of force from mechanics because he deems it to be ambiguous. And he thinks that force must be measured only by an effect, namely an observed change of motion. After all for him, it seems that force is the potential ability to change the motion of a body. Therefore, a uniformly moving body has force, and when a body is accelerated or retarded, its force increases or decreases.

He applies such a concept of force not only to a material point but also to an extended body as shown in Table 1. In addition, d'Alembert not only has different expressions for force among continuous change in motion, discontinuous change in motion and uniform motion, but also freely carries out an operation among these expressions. For example, in *Recherches sur la Précession des Equinoxes*, Lemma 5, which we discussed in §2, he integrates 'force de rotation' of an each element and 'la force φ' acting on each element. Then he adds these integrations ($\int\pi\cdot G\cdot GF+\int\varphi\cdot G$). However, form our point of view, the former expresses momentum in case of uniform rotation and the latter expresses force, so we cannot add two quantities. Fortunately, because the former is equal to zero in this case, the result does not come into question. But on the heels of this process, d'Alembert tries to get the precession of the earth by adding R and ρ to 'la force résultante' L, resulting from 'la force de rotation' of a perfect sphere as stated above. In this case, L expresses momentum and R, ρ express force. So, this operation is meaningless for us and even if his result coincides with the real precession of the earth, this agreement is nothing but accidental. However, since d'Alembert's concept of force includes both momentum and force, this operation is meaningful for him.

From the discussion above, we come to the realization that the variety in the concept of force in d'Alembert derives from a combination of the common understanding of the concept of force in his time and his unique understanding.

(* This article is based on the presentation with Dr. Jérôme Viard entitled "le mouvement de rotation d'un corps quelconque, chez d'Alembert et Euler " read at the colloquium "D'Alembert, Lalande et l'astronimie" held in September 2000. The author expresses his gratitude for the hospitality shown by Dr. Pierre Crépel to give him a chance to present his study. Additionally, the author tenders his gratitude to Dr. Jérôme Viard and Dr. Jean Souchay for discussing d'Alembert's mechanics.)

Notes

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[1] Isaac Newton, *THE PRINCIPIA Mathematical Principles of Natural Philosophy*, A New Translation by I. Bernard Cohen and Anne Whitman, (University of California Press, 1999), Definition 3, p404.

[2] *l'Encyclopédie, Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers,* éditée par d'Alembert et Diderot, art. ' Force', vol. 7, (1757), p. 110.
In this article, the following abbreviations are used: *E.O.O.*, Leonhard Euler, *Opera Omnia*; *T.D.*, D'Alembert, *Traité de Dynamique* (2nd ed. Paris, 1758); *R.P.E.* D'Alembert, *Recherches sur la Précession des Equinoxes et sur la Nutation de l'Axe de la Terre dans le Systême Newtonien* (Paris, 1749); *l'Encyclopédie, Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers,* éditée par d'Alembert et Diderot (35 vol., Paris 1751-1780).

[3] According to Euler, "Toute cause qui est capable de changer l'état d'un corps s'appelle force et partant, lorsque l'état d'un corps change, soit que du repos il commence à se mouvoir, ou qu'étant déjà en mouvement, il change ou de vitesse ou direction, ce changement vient toujours d'une force et cette force se trouve hors du corps dans quelque autre sujet, quel qu'il soit." ("Recherches sur l'origine des forces", *E.O.O.* II, 5, p.111)

[4] D'Alembert also uses the term 'la force motrice' in *T.D.* art.22 Remarque I, p.26. We will discuss this term in §.4.

[5] D'Alembert, *T.D.*, p.x..

[6] D'Alembert, art. 'Force', *l'Encyclopédie*, (1757), vol. 7, p. 111.

[7] D'Alembert, *T.D.*, p.xix.

[8] D'Alembert, *T.D.* art. 139, Lemme VIII, p.178.

[9] He gives an equation φdt=dv, where a material point changes its velocity dv during an infinitesimal time dt. Therefore, 'la force accélératrice' corresponds to acceleration.

[10] D'Alembert does not use the term 'moment of inertia'. Moreover, he always uses the velocity of each element of a body, instead of the rotational angular velocity ω.

[11] D'Alembert does not himself declare that CK signifies a particular physical quantity, he only gives the equation CK=I/(MRK·CG).

[12] As stated in §3, d'Alembert discusses a quantity corresponding to the moment of inertia but never names this quantity.

[13] D'Alembert, *T.D.*, art. 141, Collaire II, pp. 180-181.

[14] D'Alembert, *T.D.*, art. 146, Scolie I, p.191.

[15] For this reason, see Hankins, ibid., pp 217-218.

[16] D'Alembert, *T.D.*, art.142, Collaire III, pp. 181-183.
[17] As for problem 11 and 12, see, Jérôme Viard, "Le principe de D'Alembert et la conservation du "moment cinétique "d'un système de corps isolés dans le Traité de Dynamique", to be published.
[18] D'Alembert, *T.D.*, art.169, Problême XI, pp. 219.
[19] D'Alembert, *T.D.*, art.170, Problême XIII, pp. 221-222
[20] D'Alembert, *T.D.*,p.222, note 47.
[21] D'Alembert, *T.D.*, art. 172, Corollaire II, pp. 224-225.
[22] D'Alembert, *R.P.E.*, art. 86, Lemme V, p. 115.
[23] In the corollaries of this lemma, d'Alembert describes as below,
Coroll. II.
88. Soit *BMA*(Fig.38) la direction de la force qui résulte de toutes celles des points *G*, & que je nomme *F* ; je dis que $F.CM$ sera = $\int \pi . G.CG^2$. Car puisque la force *F* que je suppose agir suivant *BA* résulte de toutes les forces des points *G*, il s'ensuit que si au lieu de cette force on en supposoit une égale qui agît suivant *AB*, le corps *PKO* resteroit en équilibre, & qu'il y resteroit encore dans la même hypothese, si le point *C* ou l'Axe du corps étoit fixement attaché. Or dans ce dernier cas, la somme des momens des point *G* seroit = $\int G.\ \phi.CS + \int G.\pi.CG.CG$ = par la nature de l'équilibre, à $F.CM$. or $\int G.\ \phi.CS=0$. Donc &c.

Coroll. III.
89. Donc si le corps *PKO* est animé par les forces dont nous avons parlé dans l'énoncé du Lemme précedent, il ne pourra être retenu en équilibre que par une force *F* qui soit égale à $\int \varphi.G$, qui agisse suivant *AB* dans le Méridien *PKO* parallélement à la direction de la force φ, & enfin dont la distance *CM* au point *C*, soit = $\int \pi.G.CG^2 / \int \varphi \cdot G$.
By comparing these lemmas with corollary 2 of lemma 8 in *T.D.*, which we treated in §3, we can understand that lemma 5 in *R.P.E.* is the same problem as corollary 2 of lemma 8 in *T.D.*, if we consider that 'la force résultante *K*' in corollary 2 corresponds to F and 'la puissance *K*' to $\int \varphi \cdot G$.
[24] Be careful not to confuse d'Alembert's term 'L' with angular momentum in equation (2).
[25] D'Alembert, *R. P. E.*, art.116, Problême X, pp.141-142.
[26] D'Alembert, *R. P. E.*, art.114, Problême IX, p.137.
[27] D'Alembert, *R. P. E.*, art.115, Corollaire, p. 138.
[28] D'Alembert, *R.P.E.*, art. 116, Problême X, p. 138.
[29] D'Alembert, *R.P.E.*, art. 116, Problême X, p. 139.
[30] D'Alembert, *R.P.E.* art. 43, Problême V, p.46.
[31] D'Alembert, *R.P.E.* art. 90, Coroll. IV, p.118.
[32] "c'est [on doit estimer immédiatement la *force*] uniquement par les obstacles qu'un corps rencontre, & par la résistance que lui font ces obstacles."(D'Alembert, art. 'FORCE', *l'Encyclopédie*, (1757), vol. 7, p. 113.)
[33] "Puissance *en Méchanique,* se dit d'une force, laquelle étant appliquée à une machine, tend à produire du mouvement, soit qu'elle le produise actuellement ou non." (D'Alembert, *l'Encyclopédie*, art. 'Puissance', vol. 13, (1765), p. 556.)
[34] D'Alembert, , *l'Encyclopédie*, art. 'Puissance' vol. 13, (1765), p. 556. D'Alembert writes the similar sentence in *T.D.* (art.43 (1st ed.); art.51 (2nd ed.)), but he uses the term 'Statique', instead of the term 'Méchanique'.
[35] Hankins, ibid., p.182.
[36] D'Alembert, art.'Cause', *l'Encyclopédie*, vol. 2, (1751), pp.789 –790.
[37] In the former case, we can express force by mΔu and in the latter case, we can express force by mφ. See, §.4.
[38] D'Alembert, *T.D.*, art.22, Remarque I, p.22.
[39] D'Alembert, art. 'Percussion', *l'Encyclopédie*, vol. 12, (1756), p. 331.
[40] D'Alembert, *T.D.*, art.15-18.
[41] E.J. Aiton, "Polygons and Parabolas: Some Problem Concerning the Dynamics of Planetary Orbits", *Centaurus*, 1989, vol.31, pp.207-221, especially pp.216-218.

[42] *Der Briefwechsel von Johann I Bernoulli*, Band 2, (Birkhäuser Verlag, 1988), p.138.
[43] D'Alembert, *T.D.*, art.25, Remarque IV, p.30
[44] D'Alembert, *T.D.*art.25, Remarque IV, p.30-1.
[45] D'Alembert, *T.D.*, art.26, p.32.
[46] D'Alembert, *T.D.*, art.22, p.24.
[47] D'Alembert, art.'Dynamique', *l'Encyclopédie,* vol. 5, (1755), p.174.
[48] D'Alembert, *T.D.*, art.22, Remarque I, p.26-27.
[49] D'Alembert, art.'Dynamique', *l'Encyclopédie,* vol. 5, (1755), p.174.
[50] Craig G. Fraser, "D'Alembert's principle: the original formulation and application in Jean d'Alembert's *Traité de Dynamique*", *Centaurus, 28*, 1985, pp.31-61, especially, pp.51-55.
[51] Craig G. Fraser, "D'Alembert's principle: the original formulation and application in Jean d'Alembert's *Traité de Dynamique*", *Centaurus, 28*, 1985, pp.146-159, especially, pp.155-156.
[52] D'Alembert, *T.D.*, art.189, Corollaire II, p.254.
[53] D'Alembert, *T.D.*, art.168, Corollaire, p.218.
[54] Newton, ibid., p.111-113.
[55] Newton, ibid., p.116.
[56] D'Alembert, *T.D.*, art.22, Remarque I, p.26.