Some Remarks on Alexis Fontaine's Mechanics

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John L. Greenberg studied Alexis Fontaine's mathematics, but no one has studied his mechanics except for Montucla's mention in the following comment. In the third volume of *Hisoire des mathématiques* published in 1802, Montucla declared that Fontaine had given d'Alembert's principle in 1739. It seems that his declaration points to Fontaine's memoir entitled "Principes de l'art de résoudre les problêmes sur le mouvement des corps".[1]

For this reason, we will select this memoir to understand Fontaine's mechanics, point out some of the special features in his mechanics and finally compare his mechanics with d'Alembert's.

Fontaine's mechanics consists of 5 parts. In the first part, he explains fundamental concepts such as matter, force, motion, and so on.[2] He then discusses the motion of material points under the influence of a central force in the second part[3], the collision of bodies in the third part[4], and the motion of extended bodies in the fourth part.[5] Lastly, he discusses the motion of a center of gravity. [6]

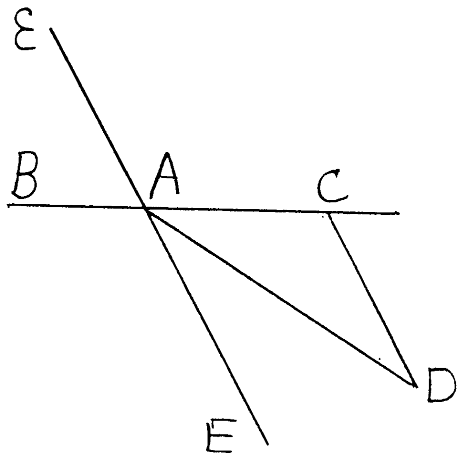
We will begin by examining Fontaine's memoir starting with the first part. First of all, he admits the existence of the vacuum and that any body will contain a vacuum in it. If a body does not contain a vacuum in it, it becomes perfectly hard. Therefore, in the discussion that follows the first part, he supposes that a body is perfectly hard, and then he discusses an elastic body with some additional conditions.

Next, with regards to force, he states: "Nous nommerons cette force *impulsion*, & nous concevrons qu'elle agit par un seul coup qu'elle frappe." Consequently, this force does not act continuously, but as an impulse without duration. His belief in the existence of a perfectly hard body or that force is impulses coincide with the notions held by d'Alembert.

As for inertia, Fontaine writes his peculiar opinion. According to him, a body has a force to maintain the status quo and this force is proportional to the mass of the body. He names this force "inertia." Consequently, inertia signifies force. This idea is common among contemporary scholars. However, he explains inertia as follows. "La force de ce corps...pour n'y être pas en l'état le plus prochain de celui où il y est, est comme sa masse." Therefore, a body at rest has a force, to say nothing of a body in motion. "La force d'un corps en repos pour ne pas se mouvoir avec une vîtesse donnée, est pareillement comme sa masse & comme cette vîtesse, & la direction de cette force est en sens contraire à la vîtesse." Accepting his declaration, it seems that a body at rest has a force of varied magnitude.

Furthermore, when the motion of the body has changed, he writes as follows (see Fig.1).
Que la force A.CD qu'a au lieu A dans la direction Aε, le corps A qui se meut dans la direction BC

Fig.1



avec une vîtesse AC pour ne pas se mouvoir dans la direction AD avec une vîtesse AD, soit vaincue subitement par une impulsion égal & oppsée, & le corps A se mouvera dans la direction AD avec une vîtesse AD.

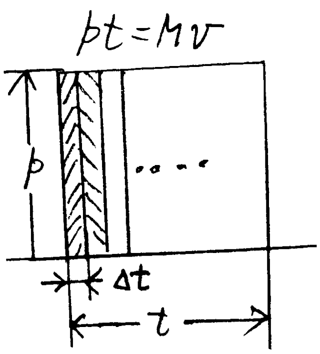
From our point of view, when a force (F) acts on a body, the body begins to move with acceleration (a) inversely proportional to a mass (m) i.e. $a=F/m$. Or using the concepts of those days, the body begins to move with velocity (v) inversely proportional to a mass (i.e. $v=F/m$) during an infinitely small time dt. Namely, when a force acts on a body, the body resists the external force by its mass, not by the product of its mass and velocity. This difference comes from the difference between the concepts of force of inertia and that of inertia. In addition, from our point of view, a bigger body can move slowly by a smaller force. We must examine in detail Fontaine's declaration that a body begins to move suddenly with a velocity *AD*.

We must add one more remark to his statement. He writes that when a body is in any motion (A), it has a force X not to be in another motion (B) (pour n'être pas à une distance v [here, v= angular velocity] de son état). We can interpret that an external force X can change the motion (A) to another motion (B). Fontaine himself states that we can rewrite the former declaration into the latter. Consequently, we will use the latter expression, though he always uses the former. However Fontaine's curious expression deserves detailed examination from a philosophical perspective.

Let us now examine the second part. He discusses the motion of material points under the influence of a central force. Here, we must treat an attraction, which acts continuously. However, since Fontaine regards any force as an impulse acting instantaneously, he must explain an attraction by impulse. We will examine his explanation.

When the velocity of a body (M: mass) changes from V to $V+v$, he interprets that a force in the body was surmounted by an external impulse Mv and this impulse acts instantaneously. Next, he replaces the impulse with 'la force motrice' as follows. "..., l'on peut supposer une cause ou *force* *motrice*, telle qu'il faudroit, pour vaincre la force *Mv*, autant de coups de cette force motrice qu'il y a d'instans dans un temps *t*." He expresses 'la force motrice' by *p*. 'La force motrice' does not have duration and acts at the beginning of each infinitely small duration *dt*. Therefore, we obtain the next relation (Fig.2)

Fig.2



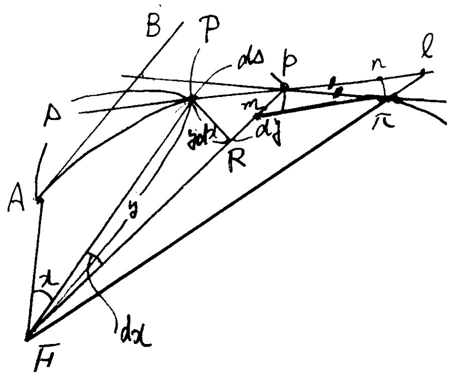
$$Mv = \sum p{\cdot}dt = p\sum dt = p{\cdot}t.$$

Consequently, force expresses *Mv* and $p=Mv/t$, which each belong to different categories. In addition, since expressing 'la force motrice' per unit mass, $p/M=v/t$ signifies "le coup frappé sur chaque point de matière de ce corps" and he names it 'force accélératrice'.

Furthermore, when discussing the relation between vis viva and 'la force motrice', he gives the relation $f=Mdv/dt$ for expressing 'la force motrice' (f) acting instantaneously. However, we must interpret this relation as differentiating the relation $Mv= p{\cdot}t$, not as the second law of motion.

After interpreting the concept of a force acting continuously as a series of impulses, he discusses the motion of material points under the influence of a central force. First, he supposes that a body moving uniformly describes a curve by receiving a series of impulses obliquely and gives the magnitude of the impulse or 'la force motrice'. Next, he supposes that the body starts from point *A* and describes curve *Ppπ*. Notations *x, y, s, dx, dy, ds* are determined as in Fig.3. *V* signifies the angular velocity of radius *FP* around point *F* and *v* signifies the velocity of *P* along the direction *FP* and *Pφ* expresses a component of 'la force motrice' along direction *F*.

Fig.3



He determines V and ds/dt, which expresses the velocity at point P as follows. During the first infinitesimal time, the body proceeds from point P to point p. During the second infinitesimal time, it continues to proceed along direction Pp to point l with "la vîtesse avec laquelle il vient de parcourir la petit côté Pp." Furthermore, the body has a little velocity to point F by 'la force motrice'. Consequently, while proceeding from point p to point l, the body proceeds from point p to point m to point F. Then, in reality, it proceeds from point p to point π, which is the diagonal of parallelogram $pl\pi m$. Therefore, he solves the problem by using the so-called d'Alembert's principle.

When solving the problem, he uses the relation $ds:pl=dx/V:dx'/V'$ or $ds:(pl-ds)=dx/V:d(dx/V)$. Since V is the angular velocity around point F and dx is an infinitely small rotational angle around point F, we obtain the relation $dx/V=dt$. The first infinitesimal time dx/V is not equal to the second infinitesimal time dx'/V', because $d(dx/V)\neq 0$. The opinion that each infinitesimal time is not equal is strange for those days.

Although we omit calculation, he demonstrates that the body describes an ellipse, of which one of the foci is point F and the areal velocity is constant. In addition, he discusses a two-body problem under the influence of a central force. Further study is necessary in order to situate in a historical context Fontaine's analyzation of the problems proposed by Newton in *Principia*.[8]

We will examine the third part, discussing collision of bodies. First, he supposes that bodies are perfectly hard and based on hard body collisions, he treats elastic body collision by adding the interaction between bodies. Here, we will treat only hard body collisions.

He regards collision as equilibrium, because bodies are perfectly hard. He writes "Les changemens qui leur [des corps qui choquent] arriveront seront tels que les forces qu'avoient ces corps pour s'y refuser, se seront vaincues mutuellement ou auront été en équilibre." A body A collides with another body B from behind. The bodies A and B have velocities a and b, respectively before collision. And since a force $A\alpha$ is surmounted by a force $B\zeta$, he obtains the relation $A\alpha+B\zeta=0$. He writes "la force $A\alpha$ du corps A dont la vîtesse est $= a$, pour que sa vîtesse ne soit pas $= a + \alpha$, ne sera vaincue que par la force $B\zeta$ qu'a le corps B, dont la vîtesse est $= b$. Pour que sa vîtesse ne soit pas $= b + \zeta$, on aura donc $A\alpha + B\zeta = 0$." These sentences are difficult to understand, but as stated above, we can interpret that after collision, the bodies A and B have velocities $a+\alpha$ and $b + \zeta$. In addition, since two bodies proceed together, due to their hardness, he obtains the relation $a+\alpha=b+\zeta$. From these relations, he gives the results $\alpha=-B(a-b)/(A+B)$ and $\zeta=A(a-b)/(A+B)$. Finally the velocity of two bodies after collision is $(Aa+Bb)/(A+B)$.

We can compare his method with d'Alembert's. According to the article 'Percussion' in *l'Encyclopédie*, he supposes that a head-on collision occurs between two bodies M and m with velocities A and a, respectively. After collision, bodies M and m have velocities V and v, respectively. He regards that velocity A consists of V and $A-V$ and that velocity a consists of v and $a-v$. He then sets the following two conditions.

1° "Les vîtesses V, v, qui sont celles que les corps gardent, doivent être telles qu'elles ne se nuisent point l'une à l'autre ; donc elles doivent être égales & en même sens, donc $V = v$".

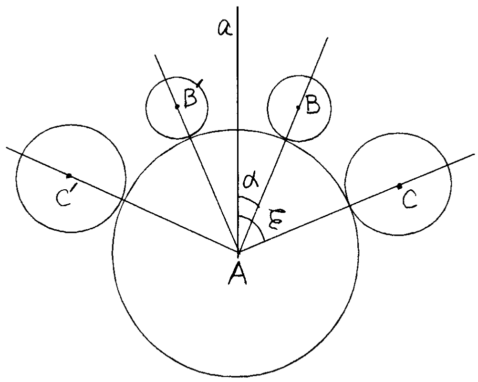
2° "Il faut que les vîtesses $A - V$, $a - v$ se détruisent mutuellement". Namely, the product of velocity A-V and mass M must be equal to the product of velocity a-v (or if considering the direction of body m after collision, $a+v$) and mass m. Therefore, he gives the relation $MA\text{-}MV=ma+mV$.

From these conditions, he gives the result $V=(MA\text{-}ma)/(M+a)$.

The second condition, in which d'Alembert uses the d'Alembert's principle, corresponds to Fontaine's equation $A\alpha + B\zeta = 0$.

After giving a law of elastic collision, Fontaine treats the next problem (Fig.4), in which a

Fig.4



sphere A collides with four hard spheres B, B', C, C' simultaneously. This problem is the same as the one d'Alembert treats in his *Traité de dynamique*[9] and *Opuscules mathématiques.*[10] They each realized that their rival discussed the same problem and mentioned each other's results. However, given the scope of this paper, we cannot make a close examination of this problem at this point.

Finally, let us now examine the fourth part, in which Fontaine treats motion of extended bodies. Here, he proposes nine problems, but we cannot examine all of them. We will examine only two problems in detail. Before solving problems, he introduces two types of forces without demonstration.

1. A weightless space has a material point A, whose distance from an axis is a. The space has a force "pour présérver son état, ou pour n'être pas en l'état le plus prochain de celui où il est". The force is proportional to the product of the inertia of the material point A (mass) and the distance from the axis a, i.e. Aa.

2. When rotating around the axis with an angular velocity V, the space has a force "pour que sa vîtesse angulaire ne soit pas $V+v$". The force is proportional to the product of the force of the space (inertia), the distance from the axis and v, i.e. Aa^2v.

Next, when the space has many material points A, B, C, D, etc., whose distance from the axis are *a, b, c, d*, etc., respectively. He then chooses one point *M*, which represents many material points in the next two relations. Here, *m* is the distance from the axis to point *M*.

$$Mm = Aa + Bb + Cc + Dd + etc.\ldots(4\text{-}1)$$

$$Mm^2v = Aa^2v + Bb^2v + Cc^2v + Dd^2v + etc.\ldots(4\text{-}2)$$

From these relations, he gives the following relations.

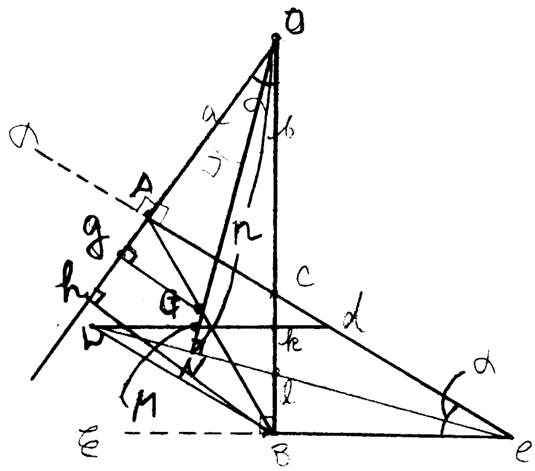
$$M = \frac{(Aa + Bb + Cc + Dd + etc.)^2}{Aa^2 + Bb^2 + Cc^2 + Dd^2 + etc.}$$

$$m = \frac{Aa^2 + Bb^2 + Cc^2 + Dd^2 + etc.}{Aa + Bb + Cc + Dd + etc.}.$$

Before discussing the physical meanings of *M* and *m*, we will examine how he solves a problem by using these quantities.

Let us first examine a problem proposed in Art.XII (Fig.5). A weightless body, which has two

Fig.5



material points *A* and *B*, can rotate around an axis *O*. He tries to determine a point *N*, i.e. to determine *ON*=*n* and *πAOG*=*v*. We will discuss the physical meaning of point *N* later. He puts *OA*=*a*, *OB*=*b*, *πAOB*=*α* and the angular velocity of *AOB*=*V*. He assumes that to increase the angular velocity as much as *v*, we must give a force *Aav* at point *A* along direction *Aα* and perpendicular to direction *OA* and a force *Bbv* at point *B* along direction *Bς* and perpendicular to direction *OB*.

To compose these two forces, he sets segments *ed* and *eB*, which satisfy the ratios *ed*:*eB*=*Bb*:*Aa*. Next he obtains *ev* by composing a parallelogram and by considering geometrical relations, he obtains the magnitude of *ev*. As a result, the resultant force along direction *ev* is

$$\sqrt{A^2a^2 + 2AaBb\cos\alpha + B^2b^2}\cdot v.$$

Next, he drops a perpendicular *ON*=*n* from the point *O* to the segment *ev* and supposes that *Nn* satisfies the following relation

$$Nn = \sqrt{A^2a^2 + 2AaBb\cos\alpha + B^2b^2}\ .$$

From the geometrical relations (not, from the equations (4-1) and (4-2)), he gives the magnitude of quantity n as

$$\frac{Aa^2 + Bb^2}{\sqrt{A^2a^2 + 2AaBb\cos\alpha + B^2b^2}}\ .$$

Finally, he obtains N,

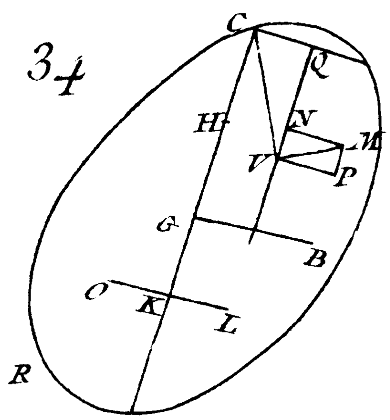
$$\frac{A^2a^2 + 2AaBb\cos\alpha + B^2b^2}{Aa^2 + Bb^2}\ .$$

He names point N "le centre de force de l'espace AOB par rapport à l'axe O."

Fontaine does not actually state this, but n justly signifies a length of equivalent simple pendulum with respect to axis O and point N expresses center of oscillation. He only refers to point N as 'le centre de force.' We can understand his nomenclature by considering d'Alembert's method[11], by which he solves essentially the same problem.

According to d'Alembert, we can regard acting force VM as an impulse acting at the beginning of an infinitesimal time dt (Fig.6). Since the velocity of point V increase dv by a force VM, we can

Fig.6



write $F_{VM}=V{\cdot}dv_V=V{\cdot}CV{\cdot}d\omega$, here ω is angular velocity and F signifies force. Since we can write $F_{VP} = F_{VM}HVQ/CV = V{\cdot}VQ{\cdot}d\omega$, we obtain the relation $F_{OL} = \int F_{VP} = \int V{\cdot}VQ{\cdot}d\omega = d\omega\int V{\cdot}VQ = d\omega{\cdot}CG{\cdot}MRC = (CG{\cdot}d\omega){\cdot}MRC = dv_G{\cdot}MRC = \varphi{\cdot}MRC$, here MRC is the mass of the body. Since the resultant force F_{OL} acts on a point K, 'le moment de la force $\varphi{\cdot}MRC$' is $\varphi{\cdot}MRC{\cdot}CK$.

Le moment de la force is the summation of elementary moment of force resulting from each point of the body and the summation gives $\int V{\cdot}(\varphi{\cdot}VC/CG){\cdot}VC$, here $\varphi{\cdot}VC/CG$ expresses the acceleration of point V by using that of the center of gravity. Therefore, we obtain the relation

$$\varphi{\cdot}MRC{\cdot}CK=\int V{\cdot}(\varphi{\cdot}VC/CG){\cdot}VC.$$

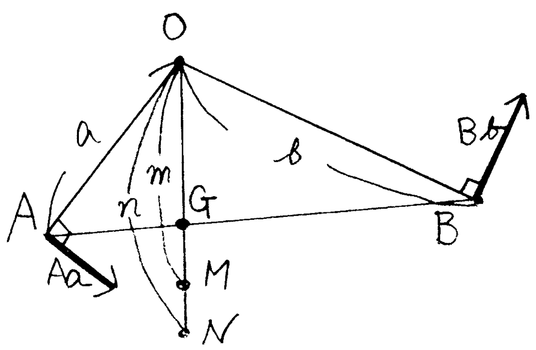
Here, *CK* signifies a length of equivalent simple pendulum with respect to axis *C*, but he does not refer to this fact either.

By comparing their methods, we can understand easily that point *N* in Fontaine's method corresponds to point *K* in d'Alembert's method and forces *Aa* and *Bb* in the former case corresponds to force *VP* in the latter case. Furthermore, force *VN* will vanish when we integrate it over the whole body. The only difference is that Fontaine considers a two material points system, while d'Alembert considers an extended body. Consequently, d'Alembert determines point *K* so as to represent a summation of elementary 'moment de la force' at each point of the body, whereas from Fontaine's point of view, point *K* can be regarded as 'le centre de force.' Here, Fontaine's force corresponds to d'Alembert's moment of force.[12] Such a complicated nomenclature was not unusual in those days.

According to Maltese, in the first half of the 18th century, scholars tried to determine special points in an extended body to explain its motion by mechanics for a system of material points.[13] For example, he cites *centrum oscillationis, centrum spontaneum rotationis*[14], *centrum virium.*[15] We can regard Fontaine's 'le centre de force' as one of the examples.

We will now examine quantities *M* and *m*, as defined in equations (4-1) and (4-2). For the sake of simplicity, we will consider them in a system of two material points (see Fig.7). Fontaine names

Fig.7



point *M* as 'le centre d'inertie *M* par rapport à l'axe *O*'. At the same time, he names the center of gravity G as 'le centre de force *G* de l'espace *AOB* par rapport à un axe infiniment distant.'

From the equations (4-1) and (4-2), we can understand that point *m* is on a circle, whose center is *O* and the radius is $(Aa^2+Bb^2)/(Aa+Bb)$, but cannot determine its position uniquely. However, Fontaine states that "le centre d'inertie *M*...est dans la ligne *OG*" and we are able to confirm this declaration by Fig.5. Furthermore, from his composition of forces acting on points *A* and *B* in his problem solved above, we can easily understand that quantity *n* signifies the distance from axis *O* to a vector, which is composed by $\overrightarrow{Aa}$ (whose origin is the point *A* and perpendicular to *OA*) and $\overrightarrow{Bb}$ (which is perpendicular to *OB*, and whose origin is point *B*). In addition, *n* satisfies the relation $Nn^2v=Aa^2v+Bb^2v$ but not $Nn=Aa+Bb$. Line *ON* passes through point *G* and *M* and

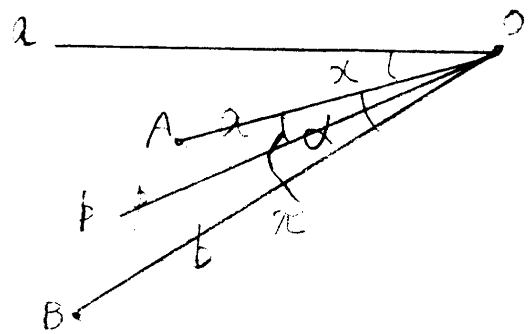
$OM=m=(Aa^2+Bb^2)/(Aa+Bb)$. We can understand that point N is a center of oscillation, but what is the physical meaning of point M?

By multiplying the relation $Mm=Aa+Bb$ by v (angular velocity), we obtain $Mmv=Aav+Bbv$ and this relation signifies that the momentum of point M represents the momentums of points A and B. However, from our point of view, although momentum is a vector quantity, Fontaine gives its relation as a scalar quantity. This confusion makes the physical meaning of M or m obscure. His relations have physical meaning only when points O, A, and B are on the same line. As a result, he uses the quantities N and n (as long as Nn^2 expresses the moment of inertia), but not the quantities M and m in solving problems.

In contrast, d'Alembert decomposes the force acting on point V (from our point of view, momentum) into two perpendicular directions; one along direction CG, the other along the direction perpendicular to CG. Next, he demonstrates that the resultant force along the direction CG is equal to zero and only the resultant force along the direction perpendicular to direction CG must be considered. In d'Alembert's resolution, the quantities m and n, which are introduced by Fontaine, coincide and point M becomes the special point on a body called the center of oscillation. Fontaine did not understand the fact that momentum is a quantity with a direction.

We will examine the next problem Art.XVI (Fig.8). D'Alembert tries to determine the motion of

Fig.8



a weightless body AOB, which has two material points A and B, when it is rotated in a vertical plane around a point O under the influence of gravity. He supposes that point A is accelerated by 'la force' $\cos x$ around axis O if point A were to exist alone. 'La force' means the acceleration of point A around axis O by setting gravitational acceleration = 1. [16]

Next, he puts 'la force' that accelerates line OA as $\cos x/a$. This force is equivalent to the acceleration of a point on line OA around axis O, whose distance from point O is a unit length. Putting the acceleration as α, we obtain the relation $\alpha dt=dv$ (v: rotational velocity). Since $v=\omega$ (ω: angular velocity) in this case, we obtain $dv=d\omega$. Therefore, we finally obtain $\alpha dt=d\omega$. Namely, at the beginning of an infinitesimal time dt, an impulse acts on the body to increase by an infinitesimal angular velocity $d\omega$ and the velocity of point A increases by $a\cdot d\omega=d(a\omega)$. [17] The situation is the same for body *B*. If it were to exist alone, body *B* is accelerated around axis *O* by 'la force' $cos\ (\alpha+x)$ and line *OB* is accelerated by 'la force' $cos\ (\alpha+x)/b$.

However, in reality, since points *A* and *B* are fixed on the body, he supposes that line *OA* is accelerated by 'la force' $cos\ x/a+e$ and line *OB* is accelerated by 'la force' $cos\ (\alpha+x)/b+f$. Consequently, by putting 'la force' accelerating body *AOB* as φ, we obtain the next relation

$$\varphi= cos\ x/a+e= cos\ (\alpha+x)/b+f.$$

When 'la force' of *OA* (from our point of view, angular acceleration) changes by *e*, 'la force' of *OA* (angular momentum) changes by Aa^2e (...la force de *OA*, pour n'être pas à une distance *e* de son état, est = Aa^2e...). And when 'la force' of *OB* (from our point of view, angular acceleration) changes by *f*, 'la force' of *OB* (angular momentum) changes by Bb^2f. Since these two forces are in equilibrium (Ces deux forces se vaincront mutuellement, ou seront en équilibre), we obtain the following relation

$$Aa^2e+ Bb^2f=0.$$

By eliminating *e* and *f* from these three equations, we obtain

$$\varphi = \frac{Aa\cos x + Bb\cos\alpha\cdot\cos x - Bb\sin\alpha\cdot\sin x}{Aa^2 + Bb^2}.$$

Since 'la force' to accelerate point *P* around axis *O* is $cos\ (\pi+x)$, 'la force' to accelerate line *OP* around axis *O* is $cos\ (\pi+x)/p$, here $OP=p$ and $\pi AOP=\pi$. Consequently, we obtain

$$\varphi= cos\ (\pi+x)/p.$$

Therefore, we obtain

$$\frac{\cos\pi\cdot\cos x - \sin\pi\cdot\sin x}{p} = \frac{Aa\cos x + Bb\cos\alpha\cdot\cos x - Bb\sin\alpha\cdot\sin x}{Aa^2 + Bb^2}.$$

By considering that *p* and π are invariable and *x* is variable, we can take the differential of the equation above with respect to *x*. From the equation obtained and the equation above, we finally accede to

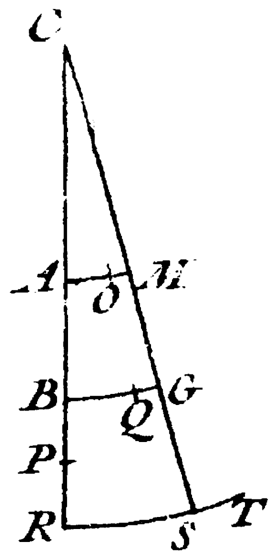
$$p = \frac{Aa^2 + Bb^2}{\sqrt{A^2a^2 + 2AaBb\cos\alpha + B^2b^2}}.$$

Fontaine refers to point *P* as follows. "Le point *P* se nomme le centre d'oscillation de l'espace *AOB*, & l'on voit qu'il est le même que le centre de force *N* de ce même espace."

Similar problems were solved by Jacob Bernoulli [18], Johann Bernoulli [19], Euler [20] and d'Alembert [21]. Christiane Vilain examines their solutions in detail.[22] Here, we will compare Fontaine's solution with d'Alembert's.

To determine the velocity of a weightless bar *CR*, which has material points at points *A*, *B* and *R*, d'Alembert tries to determine a segment *RS* (Fig.9). When the bar is on *CR*, 'vitesses' [23] *RT*, *BQ* and *AO* are given on each material point. However, in reality, being fixed on the bar, each material

Fig.9



point proceeds to *RS, BG* and *AM*. He regards the given velocities *RT, BQ* and *AO* as consisting of *AM,-MO; BG,-GQ; RS,ST*, respectively. Since 'le levier *CAR* seroit demeuré en repos, si les corps *R, B, A* n'avoient reçû que les mouvenens *ST, -GQ, -MO*' from d'Alembert's principle, he gives the next relation

$$A.MO.AC+B.GQ.BC=R.ST.CR.$$

By putting $AO=a$, $BQ=b$, $RT=c$, $CA=\gamma$, $CB=r$, $CR=\rho$, $RS=z$, he gives the value of z as follows

$$z=\frac{Aa\gamma\rho+Bbr\rho+Rc\rho^2}{A\gamma^2+Br^2+R\rho^2}.$$

In the corollary I, he puts F, f and φ as 'les forces motrices' of material points A, B and C, respectively and replaces a, b and c by F/A, f/B and φ/R, respectively.[24] Finally, he gives 'la force accélératrice' of the material point R as follows

$$\frac{F\gamma+fr+\varphi\rho}{A\gamma^2+Br^2+R\rho^2}\times\rho.$$

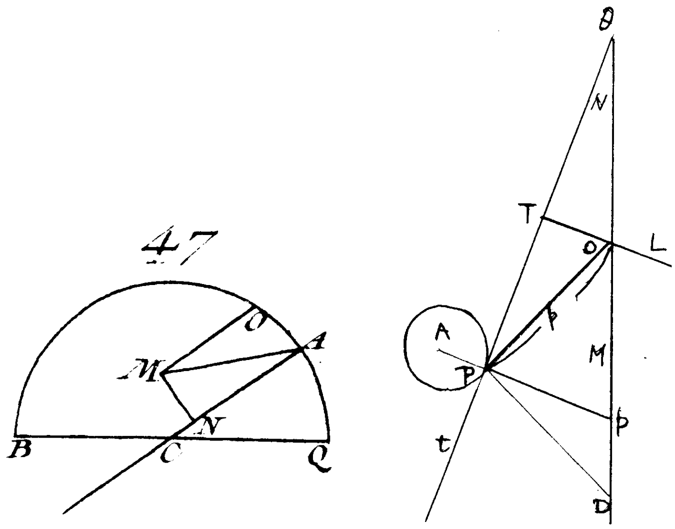
Here, $\frac{F\gamma+fr+\varphi\rho}{A\gamma^2+Br^2+R\rho^2}$ expresses the angular acceleration of the bar.

When a bar is rotating around axis C under the influence of gravity, as in Fontaine's case, $\frac{F\gamma+fr+\varphi\rho}{A\gamma^2+Br^2+R\rho^2}$ coincides with Fontaine's result for φ by setting $\alpha=0$.[25] For in this case, $F/A=f/B=\varphi/R$ are equal to a component of gravitational acceleration perpendicular to the bar CR ($=\cos x$).

Therefore, both methods are the same, in the sense that they determine the motion of an extended body by balancing the angular momentum gained and the angular momentum lost.[26]

Finally, we will quickly introduce two problems, which Fontaine and d'Alembert solve separately. The first problem is also treated by Euler[27], Daniel Bernoulli[28], d'Alembert[29]. The second problem is treated only by d'Alembert.[30] The former discusses collision between a material point and a non-fixed extended body (Fig.10) and the latter discusses collision of two extended bodies

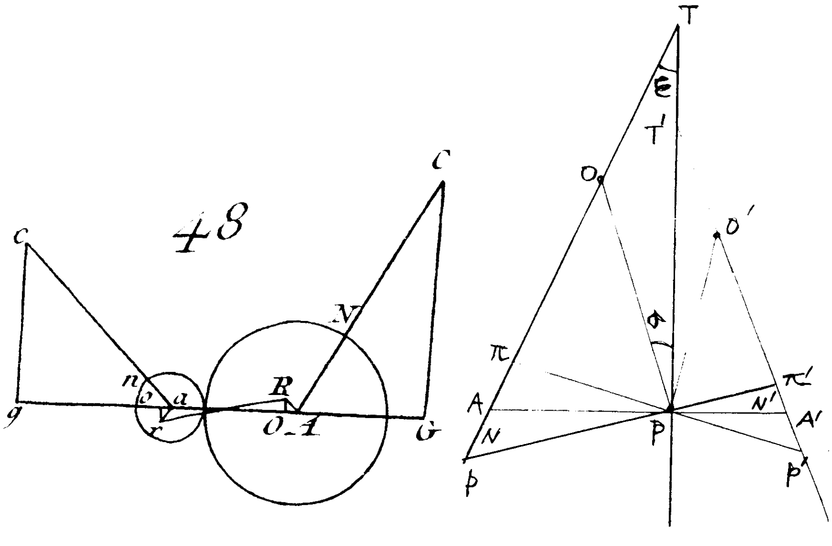
Fig.10



D'Alembert Fontaine

fixed around different axes (Fig.11). Since the solutions by d'Alembert are discussed in detail by

Fig.11



D'Alembert Fontaine

Jérôme Viard[31], we will not explain his methods any more.

Here, we will only point out what d'Alembert wrote in his solution. This will explain Fontaine's equation (4-2).

“Mais dans une Sphère qui tourne autour d'un point fixe, la quantité de mouvement & la force ne sont pas la même chose : il faut avoir égard de plus au bras de levier par lequel chaque particule agit ; c'est la somme des produits de chaque élément par sa vitesse & par sa distance au point fixe, qui fait la force, & non pas seulement la somme des produites de chaque élément par sa vitesse.”

(* This article is based on the presentation at the colloquium "Fontaine" (Manifestation soutenue par le GDR D'Alembert (CNRS), l'Université Lyon 1, les Amis de Cuisel, et parrainée par l'Académie des Sciences de Paris) held in September 2004.)

Notes

[1] Mémoires donnés à l'Académie Royale des Sciences, non imprimés dans leur temps. Par M. Fontaine, de cette Académie. –A Paris: De l'Imprimerie Royale, M.DCCLXIV.- [lbl.]f., [4]f., 588p., [lbl.]f. : [ll]pl. gr.s.c. passim ; in-4° (26, 1x20cm). pp.305-369.
However, since in his memoir, Fontaine mentioned d'Alembert's memoir published in 1761, it is not certain whether Fontaine completed the entire memoir in 1739.

[2] pp.305-313.

[3] pp.314-336.

[4] pp.336-344.

[5] pp.345-367.

[6] pp.367-369.

[7] The second part, art. XV, p.317.

[8] For example, see; Niccolo Guiccardini, “Johann Bernouilli, John Keill and the Inverse Problem of Central Forces,” *Annals of Science*, 1995, vol.52 pp.537-575.

[9] Jean Le Rond d'Alembert, *Traité de dynamique*, Remarque II, 1743, p.154, (1758, p.230).

[10] Jean Le Rond d'Alembert, *Opuscules Mathématiques*, tome I, Supplément, pp.299-303.

[11] Jean Le Rond d'Alembert, *Traité de dynamique*, Lemme VIII(1st ed., p.118 ; 2nd ed, p.178)

[12] From our point of view, this notion corresponds to that of angular momentum

[13] Giulio Maltese, “Toward the Rise of the Modern Science of Motion: the Transition from Synthetical to Analytical Mechanics,” Conference Proceedings Vol. 42, pp.51-67, “History of Physics in Europe in the 19th and 20th Centuries,” F. Bevilacqua (Ed.) SIF, Bologna, 1993.

[14] Johann Bernoulli, “Propositiones variae Mechanico-dynamicae,” *Opera omnia* 4, N°CLXXVII, pp.265-273

[15] Daniel Bernoulli, “De mutual relatione cetri virium, centri oscillationis et centri gravitates, demonstrations gemetricae,”*Commentarii academiae scientiarum Petropolitaneae* **2** 1727 (1729), pp.208-216.

[16] According to Fontaine's view, an impulse, whose duration is zero, acts on a body at the beginning of an infinitesimal time *dt*.

[17] This force signifies the angular acceleration around axis *O* from our point of view.

[18] Jacob Bernoulli, “Démonstration générale du centre de balancement ou d'oscillation, tirée de la nature du levier, ” *Mémoires de l'Académie des Sciences de Paris,* 1703, pp.78-84.

[19] Johann Bernoulli, “Meditation de Natura centri oscillationis,” 1714, Opera omnia 2, N°XCVI, pp.168-186 ; “Nouvelle theorie du centre d'oscillation,” *Mémoires de l'Académie des Sciences de Paris,* 1714, pp.208-230.

[20] Leonhard Euler, “De minimis ocsillationis corporum tam rigidorum quam flexibilium methodus nova et facilis,” *Commentarii academiae scientiarum Petropolitaneae* 7 (1734/5) 1740, pp.99-122 (*Opera omnia* ser.2, vol.10, pp.17-34.

[21] Jean Le Rond d'Alembert, *Traité de dynamique*, Problême I, 1743, p.69, (1758, p.96).

[22] Christiane Vilain, "La question du 'Centre d'oscillation' de 1703 à 1743,".

[23] From our point of view, we must write 'acceleration', not velocity. According to the note made by Bezout (p.97, note 18), '*AO, BQ, RT* sont les impulsions momentanées que les force accélératrices communiquent aux corps *A, B, R*...' Consequently, he supposes that the force accélératrice does not act continuously during infinitesimal time *dt*, but acts at the beginning of *dt* as an impulse and after that the body proceeds uniformly through time *dt*. In those days, the difference between moment of force and moment of moment (angular momentum) was obscure, because for mechanicians in those days, force meant momentum.

[24] a, b and c signify the distance traveled during an infinitesimal time, i.e. velocities. By putting a, b and c as F/A, f/B and φ/R, we can understand that 'les forces' F, f and φ express momentum.

[25] We suppose that two material points are at points A and R, as Fontaine assumes. By putting $CB=r$ as the length of equivalent simple pendulum and $BG=b$, from d'Alembert's principle we obtain the relation $R\rho(c-b/rH\rho)=A\gamma(b/rH\gamma-a)$. Solving the relation with respect to R, we obtain $r=(A\gamma^2+R\rho^2)/(R\rho Hc/b+A\gamma Ha/b)$. When the bar rotates in the vertical plane under the influence of gravity, we can put $a=b=c$. Therefore, we obtain the result $r=(A\gamma^2+R\rho^2)/(R\rho+A\gamma)$.

[26] D'Alembert writes motion and Fontaine writes force.

[27] Leonhard Euler, "De communicatione motus in collisione corporum sese non directe percutientium," *Commentarii academiae scientiarum Petropolitaneae 9* (1737) 1744, pp.50-76, (*Opera omnia* ser.2, vol.8, pp.7-26.

[28] Daniel Bernoulli, "De variatione motuum a percussione excentrica," *Commentarii academiae scientiarum Petropolitaneae 9* (1737) 1744, pp.189-206.

[29] Jean Le Rond d'Alembert, *Traite de dynamique*, Problême XI, 1743, p.145, (1758, p.219).

[30] Jean Le Rond d'Alembert, *Traite de dynamique*, Problême XII, 1743, p.147, (1758, p.221).

[31] Jérôme Viard, "Le principe d'Alembert et la conservation du 'moment cinétique' d'un système de corps isolés dans le traité de dynamique," *Physis*, vol.XXXIX, 2002, pp.1-40.