Analysis of motion of a rotating tube including a material point by Johann Bernoulli, Daniel Bernoulli, Clairaut, d'Alembert and Euler

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§1. Introduction.

One of the most attractive problems in 1740s was a motion of a rotating tube around a fixed point and that of a small body included in it. Representative mechanicians in those days, such as Daniel Bernoulli, Alexis-Claude Clairaut, Leonhard Euler and Jean le Rond d'Alembert, tried to solve this problem.[1] This problem was later applied to astronomical mechanics. It was posed by Euler for Johann Bernoulli in his letter in 1741.[2] According to Johann Bernoulli's letter to Euler dated on March 15, 1742, this problem is to determine a motion of a rotating tube around a fixed point and that of a body included in it under the influence of the gravity.

> Sit tubus seu canalis (sive gravis sive gravitatis expers) mobilis circa axem fixum, in quo versetur globus, qui ob gravitatem in tubo sine frictione descendat (et quidem, quod sine dubio subintelligis, non rotando sed fluendo) simulque tubo motum inducat : quovis tempore determinare situm tubi et globi in tubo, itemque utriusque celeritatem.[3]

Johann Bernoulli tried to solve this problem in his succeeding letter but he failed to solve it.[4]

Johann Bernoulli in reality solved a problem posed by J.S. König in autumn of 1743 when König studied with Clairaut and Maupertuis.[5] It is as follows,

> Determiner la courbe, que decrit un corps renfermé dans un Tuyeau pendant que le tuyeau se meut uniformement autour d'un Centre sur un plan horizontal.[6]

According to Johann Bernoulli's letter to Euler dated on August 27, 1742, these two problems seem to be similar but for him, they were quite different. He could solve the problem posed by König but the problem posed by Euler was unsolvable for him. He admitted that there was a great disparity between them and summarized difference as follows. 1. König considers a case where the gravity does not exist (a motion on a horizontal plane) but Euler poses motion under the influence of gravity. 2.A tube rotates uniformly in the former case but it does not rotate uniformly in the latter case. 3. Motion of a body included in a tube depends on the tube in the former case but in the latter case, a tube is weightless and its motion depends on a heavy body.

Johann Bernoulli published the resolution of this problem twice. First, he wrote it in French as a supplement in his letter to Euler dated on August 27, 1742[7] and after that, he rewrote and published it[8] in the 4th volume of his complete works in Latin.[9]

Euler also posed this problem to Daniel Bernoulli and Clairaut. Daniel Bernoulli frequently mentioned it in a series of letters to Euler.[10] Especially, in his letter dated on October 20, 1742,[11] he discussed a motion of a heavy straight tube rotating around a fixed point on a horizontal plane and that of a material point included in it. Next, he solved this problem by using the principle of conservation of "le momentum du mouvement circulatoire", which corresponds to the conservation of angular momentum for us, and published it in his memoir.[12]

Clairaut also gave an outline of this resolution in his letter to Euler[13] and published it in his memoir.[14] Euler himself published his resolution in his memoir.[15] D'Alembert alone discussed this problem in his book independently.[16]

In this article, we will examine Johann Bernoulli's and Daniel Bernoulli's methods to solve this problem in §.2 and 3. In §. 4, 5 and 6, we will summarize Clairaut's, d'Alembert's and Euler's methods. Finally, we will compare and examine their five methods and discuss their positions in the 18th century mechanics, which was on the way to constitute the modern physics.

§.2 Motions of a rotating tube including a body – Johann Bernoulli's analysis.

Johann Bernoulli's two methods are essentially the same. Then, we will employ the French version here.

First, he introduces a lemma (Fig.1).

Fig.1
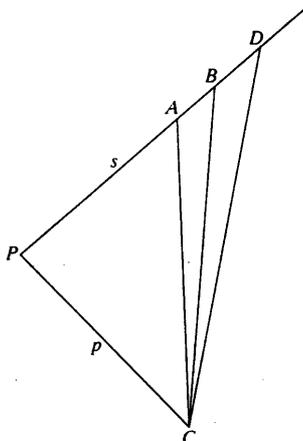
Lemme, Si d'un point *C* l'on tire les droites, *CA, CB, CD*, qui coupent sur *PD* les parties *AB*, *BD*, infiniment petites et égales, la différence des angles *ACB* et *BCD* c'est à dire

ACB-BCD=(2psds)²/(pp+ss)²...(2-1)

en nommant la constante *CP=p*, perpendiculaire sur la droite variable *PA=s*. Cela se démontre facilement en différentiant l'angle *ACB* dans la supposition de *ds* constant.

We can demonstrate the equation (2-1) as below. By putting $\pi PCA=\theta$, we obtain $\pi ACB=d\theta$ and by differentiating an equation $\cos\theta=p/o(p^2+s^2)$ we obtain $-\sin\theta \cdot d\theta = -1/2 \cdot (p^2+s^2)^{-3/2} p \cdot 2sds$. Substituting

$\sin\theta = s / \sqrt{p^2 + s^2}$ for this equation and solving it for dθ, the resolution is $d\theta = \dfrac{pds}{p^2 + s^2}$.

Then, by differentiating it once more under the condition of ds=constant, we can obtain

$$\left|\angle ACB - \angle BCD\right| = \frac{2psds^2}{\left(p^2 + s^2\right)^2}.$$

Next, he solves a problem.

Probleme. Déterminer la courbe que décrit un corps renfermé dans un tuyau pendent que le tuyau se meut uniformément autour d'un centre sur un plan horizontal.

In Fig.2[17], *ABE* represents a curve described by a body when a tube *CA* rotates around a point *C*. We

Fig.2
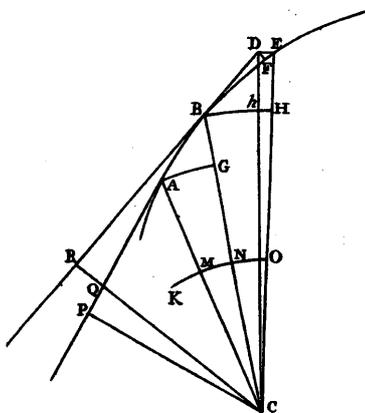
put *CA=x, AB=ds* and *BH=dy*. The body describes an infinitesimal line *AB* during an infinitesimal time. If it were not restricted, it would describe another infinitesimal line *BD* (*=AB*), which is a prolonged line of *AB* during a succeeding equal infinitesimal time. But the tube proceeds to *CE* at

the end of the second infinitesimal time and πECB is equal to πBCA. Then, in reality the body proceeds to a point E. Here, the tube rotates equal angles during two equal infinitesimal durations.

He drops perpendiculars CP and CR to lines BP and ER, respectively and since πCAG is equal to πCGA and πR, because of $CA=CG$, he obtains the relation that triangle CPA is similar to triangle AGB. Then he can obtain $p = CP = x \cos \angle ACP = x \cos \angle BAG = x\,dy/ds$. Similarly he obtains the relation $PA = x\,dx/ds$, therefore, $\left| \angle ACB - \angle BCD \right| = 2dydx/xx$. Furthermore, he obtains

$$\frac{d\left(xdy/ds\right)}{xdx/ds} = \frac{DF}{ds}$$

,because of $RQ/QB=DF/BF$ and from this,

$$DF = \frac{ds^2 \cdot d\left(xdy/ds\right)}{xdx}.$$

Next, he obtains

$$dx.ds :: \frac{ds^2 \cdot d\left(xdy/ds\right)}{xdx}.DE ,$$

because triangle HDB is similar to triangle FDE. By considering that ds is constant, he obtains

$$DE = \frac{ds^2 \cdot d(xdy)}{xdx^2}.$$

Therefore, he gives the relation

$$\frac{DE}{EC} = \frac{ds^2 \cdot d\left(xdy\right)}{xxdx^2} = \angle DCE .[18]$$

From the relation $\in DCE = \in BCA - \in DCB$, he obtains

$$xdx^2 ddy + xdy^2 ddy = dydx^2 - dxdy^3$$

and by transforming this relation into

$$xdx^2 ddy - xdy^2 ddy + 2xdy^2 ddy = dydx \cdot \left(dx^2 - dy^2\right),$$

he obtains

$$\frac{ddy}{dy} + \frac{2dyddy}{dx^2 - dy^2} = \frac{dx}{x}.$$

By considering $dx^2 - dy^2 = ds^2 - 2dy^2$, this relation can be transformed into

$$\frac{ddy}{dy} + \frac{2dyddy}{ds^2 - 2dy^2} = \frac{dx}{x}.$$

By integrating it, he obtains

$$l\left(ndy\right) - 1/2\,l\left(ds^2 - 2dy^2\right) = lx ,[19]$$

where n is a constant.
From this relation, he gives

$$\frac{ndy}{\sqrt{dx^2 - dy^2}} = x$$

or

$$dy = \frac{xdx}{\sqrt{nn + xx}}.$$

By replacing dy with a and dz, which are equal to a radius CM and an infinitesimal element of an arc MN, respectively, he obtains the relation

$$dz = \frac{adx}{\sqrt{nn+xx}}.$$

By integration, he reaches at the final relation

$$z = l\left(x + \sqrt{nn+xx}\right).$$

This equation represents a curve described by the body in the tube.[20]

§.3 Motions of a rotating tube including a body – Daniel Bernoulli's analysis.

§.3-1 Analysis in the letter to Euler in 1742 (Daniel Bernoulli [a]).[21]

Daniel Bernoulli, in his letter to Euler dated on December 20, 1742, wrote that "Ew. Problema generalissimum circa motum globi in tubo hab ich auch solvirt :.." then, we will examine his solution.

In Fig.3, when a tube AD including a small sphere rotates around a point A on a horizontal

Fig.3
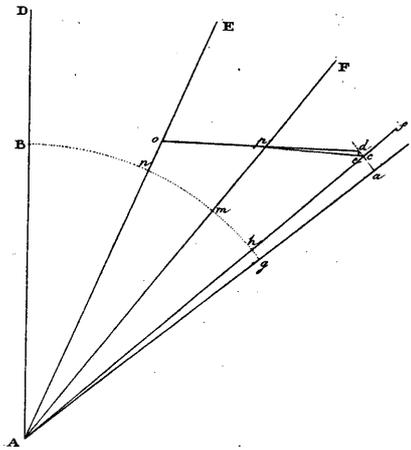
plane, he tries to determine velocities of the tube and the sphere and a curve described by the sphere.

Moveatur tubus AD continens globum F super plano horizontali circa polum A, sitque determinanda curva, quam describet globus una cum velocitatibus globi et tubi.[22]

At the first instant, the tube and the sphere are at rest at AD and B, respectively. Then, a force acts on the tube perpendicularly and after that, no external force acts on it. A curve Bnm represents an arc described by the point B around the point A. Now, the tube and the sphere proceed to AE and a point a, respectively. After an infinitesimal instant dt, the tube and the sphere go to AF and a point p, respectively, and op represents an infinitesimal line described by the sphere during dt. He supposes that after arriving at the point p, the sphere is free from constraint of the tube, then it will describe an infinitesimal prolonged line pd ($=op$) during a succeeding time dt. And the tube will proceed to Ab without interaction. Here, an arc nm is equal to an arc mg. By drawing an arc ac and setting $AB=a$, $Bn=x$ and $Ao=y$, he obtains the relation $da = 2dxdy/a$ [23]. However, in reality, the tube and the sphere interact during the second instant dt. To explain this situation, he assumes a force (potentia) and the sphere is pressed by it against the tube and another force acts on the tube at the point a and presses it against the tube. As a result, they will meet at a point c.[24] Namely, in reality, the tube and the sphere will be Acf and the point c, respectively after the second infinitesimal instant dt. Assuming that m stands for a mass of the sphere, M for a mass of the tube, d for a distance from the point A to the center of gravity of the tube, D for a distance from the point A to center of oscillation, he obtains "erit ex mechanicis $ac : dc = m : \dfrac{dD}{yy}M$ [25]." From these equations, he obtains

$$ac = \frac{myy}{myy + MdD} \cdot \frac{2dxdy}{a} \quad \text{and} \quad dc = \frac{MdD}{myy + MdD} \cdot \frac{2dxdy}{a}$$. Next, by drawing a line de

perpendicular to a line pc, he obtains $ec = \dfrac{MdD}{myy + MdD} \cdot \dfrac{2dxdy}{a} \cdot \dfrac{ydx}{ads} = dds$.[26] Here, he puts op

as ds. He also obtains $hg = \dfrac{my}{myy + MdD} \cdot 2dxdy = -ddx$. By assuming that a velocity of the

point B is c and that of the point n is V and from the equation $ddx = dV \cdot dx/V$, because of $dt=dx/V$,

and the relation for hg, he obtain the relation $\dfrac{2mydy}{myy + MdD} = -\dfrac{dV}{V}$. And integration of this

relation gives $V = \dfrac{maa + MdD}{myy + MdD} \cdot c$. Therefore, he obtains $dx = Vdt = \dfrac{maa + MdD}{myy + MdD} \cdot cdt$ and

substituting the relation for ec, this relation gives $\dfrac{2ydy}{(myy + MdD)^3} = \dfrac{aadsdds}{MdD(maa + MdD)^2 ccdt^2}$.

By integrating this relation, he obtains $\dfrac{-1}{2m(myy + MdD)^2} = \dfrac{aads^2}{2MdD(maa + MdD)^2 ccdt^2} - C$,

here C is a constant. Representing a absolute velocity of the sphere at the point o by u and supposing that at the point B, u is equal to c and the sphere does not have any velocity component along the direction AD, because of $dt = ds/u$, he obtains

$$u = c\sqrt{\frac{maa + MdD}{maa} - \frac{MdD}{maa}\left(\frac{maa + MdD}{myy + MdD}\right)^2}$$.

A equation of a curve described by the sphere is given by the relation $V : u = dx : ds$.

Finally, he gives a pressure of the sphere caused by the tube by $\dfrac{4VV}{a} \cdot \dfrac{dy}{dx} \cdot \dfrac{MdD}{myy + MdD} \cdot m$ [27]

§.3-2 Analysis in his "Nouveau Probleme de Mécanique" (Daniel Bernoulli [b])

In this article, Daniel Bernoulli defines a circular motion of a body as le mouvement circulaire and its velocity along a tangent to the circle as la vîtesse circulatoire in §.I. In §.II, he defines a normal motion caused by circular one as le mouvement centrifuge and its velocity along a normal direction as la vîtesse centrifuge. In §.III, he defines the product of la vîtesse circulatoire, a mass of the body and a distance from the body to the center of rotation as le momentum du mouvement circulatoire. This quantity corresponds to angular momentum. In §.IV, he shows that even a body proceeding on a straight line BD, has le mouvement circulaire and le mouvement centrifuge, if we consider it from a point A (Fig.4)

Fig.4
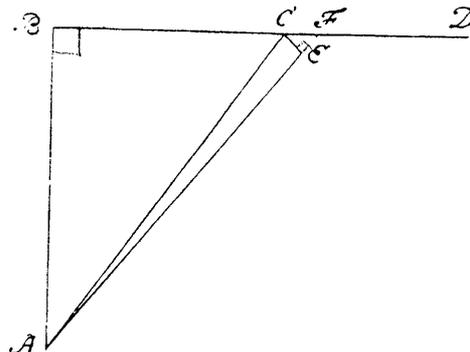
Next, he considers a tube as an extended body and he describes that le momentum du mouvement circulatoire of a tube rotating uniformly around a fixed point is conserved in §.V and gives its quantity in §.VI. Representing a distance from a fixed point to an given point of the tube by y, la vîtesse circulatoire of the given point by V, a distance from the fixed point to an any point of the tube by x, an infinitesimal element of a distance x by dx and an infinitesimal mass element of the tube by $d\xi$, he gives le momentum du mouvement circulatoire by $V/y \times \int xx d\xi$ or $MdDV/y$.

Moreover, in §.VII, he introduces an important lemma.
Si l'on applique au tuyau mobile autour d'un point fixe une puissance dans un point quelconque dont la distance au point fixe soit = y, cette puissance produira la même acceleration ou le même retardement sur le tuyau, qu'elle produiroit si le tuyau n'avoit
point de masse & qu'il y eut dans le point, où la puissance est appliquée, une masse concentrée, qui fut

$$= \frac{dD}{yy} \times M^{28}$$

He wrote in §.IX, Proposition fondamentale as below.
Si un tuyau droit renfermant un globe librement mobile tourne sur un plan horizontal
autour d'un point fixe, je dis qu'il y aura toujours le même *momentum* du mouvement circulatoire dans le Systeme du tuyau & du globe.
A sphere is included at a point B, when a tube has a position AD (Fig.5). At any instant, the tube and

Fig.5
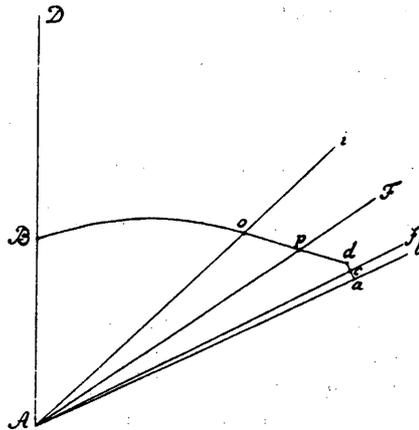
the sphere are at Ai and at a point o, respectively. To determine motions of the tube and the sphere, he considers them as follows. He assumes that during an infinitesimal time dt, the tube proceeds from Ai to AF and the sphere proceeds from the point o to a point p and during the succeeding infinitesimal time dt, if the sphere becomes free from constraint of the tube, the tube and the sphere will proceed from AF to Ab and the point p to a point d, respectively. Consequently, πFAb is equal to πiAF ,and op and pd are equal and on a straight line. From §.IV and V, he concludes that le momentum du mouvement circulatoire of the tube and the sphere are conserved respectively during free motion.
However, in reality, the tube and the sphere interact during the second time dt.
...or cette action consiste à reünir le globe d avec le tuyau Ab, & comme cette action ne sçauroit se faire que perpendiculairement au tuyau, on tirera la petite da perpendiculaire à Ab, & puis on concevra cette da comme un fil attaché par ses deux bouts au globe & au tuyau, qui se resserre entierement jusqu'à ce que les deux bouts viennent à se toucher au point c, par là on voit qu'au bout du second element de tems la vraie situation du globe
sera en c & celle du tuyau en Acf ;...
The editor of the complete works of Daniel Bernoulli named this method the 'Daniel Bernoulli-

d'Alembert's principle'. By putting $Ac=y$ and supposing that the others have the same meaning as §.3-1, constriction of a string da has the same effect when the tube is weightless and instead has a concentrated mass $dD/yy \times M$ at the point a based on the discussion in §.VIII. Therefore, he obtains the same relation as in §.3-1,[29]

$$dc : ac = dD/yy \times M : m.$$

Consequently, he describes that the increase of le momentum du mouvement circulatoire of the sphere is equal to the decrease of le momentum du mouvement circulatoire of the tube and that le momentum du mouvement circulatoire of the whole system has the same value at each instant.

Moreover, he proves that the increase of la vîtesse circulatoire of a body during dt $(=dv)$ are represented by

$$dv = \frac{VV}{y} \times dt$$

in both cases, where the body moves uniformly (§.XII) or under the influence of central force (§XIII). Here, V represents la vîtesse circulatoire of the body at that instant and y represents a distance from a fixed point in §.XII and a distance from a center of force in §.XIII. Consequently, in §.XIV, he demonstrates that when a straight tube including bodies in it rotates around a fixed point, the increase of la vîtesse circulatoire of each body is $\dfrac{VVdt}{y}$.

Then, he transfers to Problème principal in §.XIV.

Soit sur un plan horizontal un tuyau droit AF renfermant tant de Corps qu'on voudra en B, C &c. & soit d'abord tout le systeme en repos ; qu'on s'imagine ensuite que le tuyau recoive une impulsion & commence à tourner avec une vîtesse donnée autour du point A ; il est question de determiner à chaque moment & à chaque situation du tuyau le mouvement de toutes les parties du systeme.

In Fig.6, small bodies included in a tube proceed from a point E to a point l and a point D to

Fig.6
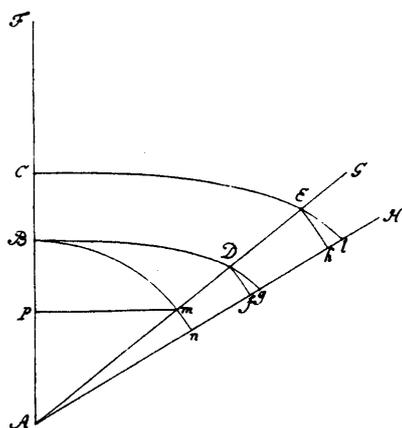
a point g, respectively during an infinitesimal time dt. Eh, Df and Bmn represent arcs and m, mN...represent masses of each body B, C,...respectively. Moreover, p, pN...represent les vîtesses centrifuges of each body, a, aN...represent distances of each body from a point A to each body on AF and x, xN...represent distances of each body from the point A to each body on Ag. In addition, y standts for an arc Bm, dy stands for an arc mn[30] and C stands for la vîtesse circulatoire at a point B, V stands for la vîtesse circulatoire at a point m.

Le momentum du mouvement circulatoire of a whole system in the first instant is equal to

$$\frac{MdD + maa + m'a'a' + ...}{a} \cdot C$$

and when the tube proceeds to *ADEG*, le momentum du mouvement circulatoire of the whole system is equal to

$$\frac{MdD + mxx + m'x'x' + \dots}{a} \cdot V.$$

Because le momentum du mouvement circulatoire of the whole system is conserved, he obtains

$$V = \frac{MdD + maa + m'a'a' + \dots}{MdD + mxx + m'x'x' + \dots} \cdot C.$$

From §.XIV, he obtains

$$dp = \frac{xVV}{aa} dt$$

and because of $p = dx/dt$, he obtains

$$aapdp = VVxdx.$$

Substituting the value of V for this equation, he gives

$$aapdp = \left(\frac{MdD + maa + m'a'a' + \dots}{MdD + mxx + m'x'x' + \dots} \right)^2 CCxdx.$$

In addition, substituting

$$x' = \frac{a'}{a} x, \ x'' = \frac{a''}{a} x, \dots^{31}$$

for the equation above and integrating it, he obtains

$$pp = \frac{(MdD + maa + m'a'a' + \dots) \times (xx - aa)}{MaadD + (maa + m'a'a' + \dots)xx} CC.$$

Les vîtesses centrifuges of each body, p, pN... are determined by

$$p' = \frac{a'}{a} p, \ p'' = \frac{a''}{a} p, \dots$$

Then, he obtains

$$V = \frac{MdD + maa + m'a'a' + \dots}{MaadD + (maa + m'a'a' + \dots)xx} aaC.$$

and gives a time, which the tube proceeds from ABCF to ADEG, by

$$\int \frac{dx}{p}.$$

Finally, by using two relations $ma + mNaN + \dots = s\delta$ and $ma^2 + mNaN^2 + \dots = \epsilon\delta s$, $(s = m + mN \dots : \delta =$ a distance from the center of gravity to the point A: $\epsilon =$ a distance from the center of oscillation to the point A), he transforms the three equations derived above as bellow.

$$pp = \frac{(MdD + s\delta\Delta) \times (xx - aa)}{MdDaa + s\delta\Delta xx} CC$$

$$dy = aadx \times \frac{\sqrt{MdD + s\delta\Delta}}{\sqrt{(MdDaa + s\delta\Delta xx)(xx - aa)}}$$

$$dt = \frac{dx\sqrt{MdDaa + s\delta\Delta xx}}{c\sqrt{(MdD + s\delta\Delta)(xx - aa)}}$$

Daniel Bernoulli solves these three equations in two extreme cases. The former is that the mass of the whole bodies can be neglected by comparing with that of the tube[32] and the latter is that conversely mass of the tube is negligible by comparing with that of the whole bodies. Finally, he finishes his memoir by describing the conservation of vis viva in general.

§.4 Motions of a rotating tube including a body – Clairaut's analysis.

Clairaut discusses motion of a tube including a body in his memoir.[33] In §.XXIII, he poses the following problem (Fig.7) and solves it.

Fig.7
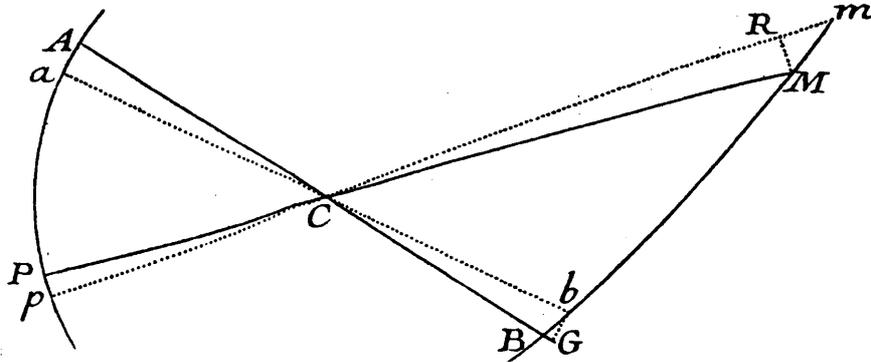
Soient sur un plan horizontal deux poids P & M, attachez à une ligne inflexible PCM, mobile autour du centre C; le premier des deux poids étant fixé sur la ligne inflexible, & l'autre pouvant glisser, on demande la courbe que décrit le corps M lorsqu'on donne une impulsion quelconque au corps P.

He assumes a system constructed by a weightless rod and two material points places at the both ends of the rod on a horizontal plane. One of the material points is fixed at one of the ends, the other is movable on the rod and the rod can rotate around a fixed point on the rod. First in §.XXIII-XXVIII, he discusses motions of the rod and the movable material point after an impact acted on the fixed material point. Next in §.XXIX-XXXII, he discusses a case, where the rod rotates on a vertical plane. In §.XXIII-XXXVII, and in §.XXVIII, he discusses cases, where the rod with two movable material points rotates on a horizontal plane and on a vertical plane, respectively. In §.XXXIX, he discusses a case, where the rod with any number of material points rotates on a horizontal plane and finally, in §.XL, he discusses a case, where the rod with any number of material points moves freely, not around a fixed point. Here, it is sufficient to examine the case discussed in §.XXIII-XXVIII. Clairaut gives two methods to solve this case. Let us examine his methods briefly.

§.4-1. Solution using the principles of la forces accélératrices and the conservation of vis viva (Clairaut [a]).

He puts $CP=1$, $Pp=dx$, $CM=y$ and $Rm=dy$ and represents an infinitesimal duration, which the rod proceeds from P to p or from M to m, by dt, mass of movable material point M by 1 and mass of fixed material point P by m, respectively. Then, a velocity of the body M along the direction CM is equal to dy/dt and its velocity along the direction MR is equal to $y \cdot dx/dt$.

First, because la force accélératrice of the body M along the direction CM is equal to $y \cdot dx^2/dt^2$, "par le principe général des forces accélératrices" he obtains

$$\left(\frac{ydx^2}{dt^2}\right)dy = \frac{dy}{dt} \times \frac{ddy}{dt}. \text{ [34]}$$

Therefore, he obtains

$$ddy = ydx^2 \dots(4\text{-}1\text{-}1).$$

Next, from the principle of conservation of vis viva, he gives

$$yydx^2 + dy^2 + mdx^2 = adt^2 \text{ (4-1-2).[35]}$$

By differentiating (4-1-2) and substituting (4-1-1), he obtains

$$(m + yy)ddx + 2ydydx = 0$$

and by integrating, he obtains

$$(m + yy)dx = bdt .$$

By using (4-1-2), he finally obtains

$$dx = \frac{dy}{\sqrt{(m + yy)}\sqrt{[(yy + m)a/bb - 1]}} .$$

§.4-2. Solution using another principle (Clairaut [b)]).

Clairaut states the principle used here as below.

ARTICLE IV.
PRICIPE GÉNÉRAL ET DIRECT

Pour résoudre le Problèmes où il s'agit de déterminer le Mouvement de plusieurs Corps qui agissent les uns sur les autres, soit par des fils, soit par des leviers, soit de toute autre manière qu'on voudra.

§. XXII.

Je commence par imaginer le système dans une situation quelconque, & je trace chacune des petites droites que les corps parcourent dans un instant ; je place ensuite au bout de ces petites droites, celles que les mêmes corps décriroient l'instant d'après s'ils étoient libres : cela fait, je marque sur les directions suivant lesquelles les fils, les leviers ou autres instrumens agissent, de petites droites qui doivent exprimer les forces de ces instrumens, & que je détermine par cette condition que les diagonales des parallélogrammes faits sur ces petites droites & sur les prolongemnes des côtés parcourus par les corps dans le premier instant, soient terminées par des points où les corps étant supposez dans le second instant, les fils ou les leviers n'auroient souffert ni extension ni inflexion. Ayant par cette méthode deux côtés consécutifs quelconques de chacune des courbes décrites par les différens corps qui composent le système donné, la manière de trouver les équations de ces courbes n'est plus qu'une affaire de calcul. Ce principe, ainsi que les précédens, sera éclairci dans l'article suivant.[36]

Meli named this principle 'd'Alembert's principle'.[37] Before solving this problem, he proposes two lemmas as preparation.

LEMME I.

Mm & mn *étant deux droites infiniment petites & égales, prise sur la droite* HO ; MC, mC, nC *trois droites tirées au point fixe* C, *je dis que si on nomme* CM, y ; Cm, y+dy ; & *l'angle* MCm, dx ; *on aura l'angle* mCn=dx-2dydx/y.[38]

We can demonstrate this lemma as follows (Fig.8). Because triangle MRm is congruent to

Fig.8
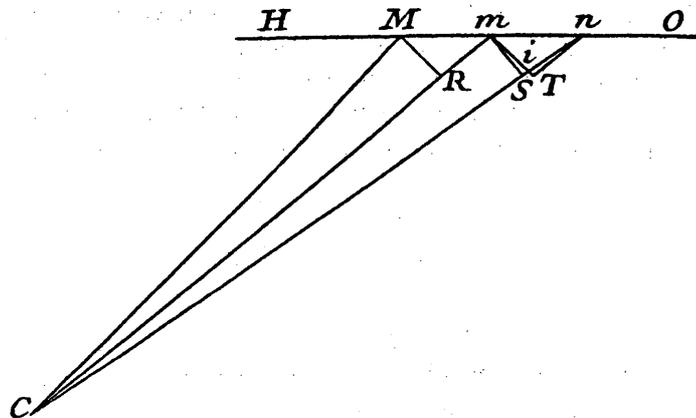
triangle mTn, we get relations MR=y·sin dx.y·dx=mT, mR=y+dy-v(y^2-y^2dx^2).dy+y/2·dx^2.dy. And

because triangle iCm is similar to triangle inT, we obtain the ratio (y+dy) :dy=(ydx-iT) :iT. By ignoring the infinitesimal smaller than the third order, we get the relation iT(y+2dy)=ydxdy. Finally, we obtain iT=ydxdy/(y+2dy)=ydxdy·1/y·(1+2dy/y)$^{-1}$.dxdy(1-2dy/y).dxdy. Therefore, we obtain mi=mT-iT=ydx-dxdy. Then, we reach the final relation πmCn=(ydx-dxdy)/[y(1+dy/y)].(ydx-dxdy)/y×(1-dy/y)=1/y·(ydx-2dxdy+dxdy^2/y).dx-2dxdy/y, because of tan πmCn.πmCn=(ydx-dxdy)/(y+dy). Here, we ignore the infinitesimal smaller than the third order.

<div align="center">LEMME II.</div>

Les même chose étant posées, je dis que Cn=y+2dy+ydx^2.[39]

We can demonstrate this lemma as follows. As stated above, triangle MRm is congruent to triangle mTnm, we obtain nT=mR. Because we can regard that nT is equal to ni, we get the relation iS=nS-nT. Furthermore, we obtain iS=mS^2/CS because triangle CSm is similar to triangle mSi. Next, we get the relation iS=y^{2}dx^2/(y+dy).ydx^2(1-dy/y)=ydx^2-dx^{2}dy.ydx^2, because of mS=πmCS×C=(dx-dxdy/y)(y+dy)=ydx-dxdy^2/y.ydx, here we ignore the infinitesimal smaller than the third order. Finally, we obtain the relation Cn=CS+iS+in=y+dy+ydx^2+dy=y+2dy+ydx^2.

Next, in §.XXVIII, he solves the problem. In Fig.9, a rod rotates around a point *C* and

<div align="center">Fig.9</div>
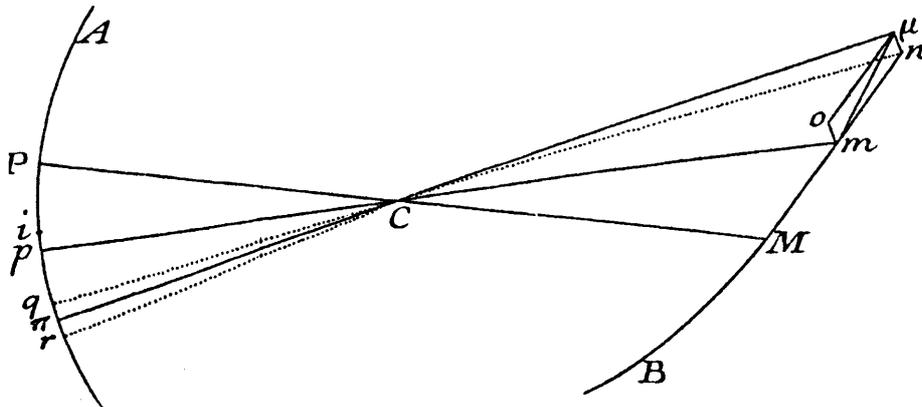
proceeds to *PCM* from *ACB*. *P* and *M* stand for material points and *P* is fixed on the rod but *M* is movable on the rod. He puts *CP=1* and *CM=y* and supposes that *P* proceeds to *p* and *M* proceeds to *m* during the infinitesimal time *dt* and puts *πPCp=πMCm=dx*. During the succeeding time *dt*, he supposes that if two bodies does not interact, the body *P* will describe *pr* and the body *M* will describe *mn*, here *r* is on an arc *AP* and *mn* is on the prolonged line *Mm*. However, in reality, bodies are constrained by the inflexible rod and he represents this constraint to the body *P* and to the body *M* by impacts *pi* and mo, respectively. These impacts are perpendicular with respect to the rod *mCp* and have opposite directions mutually. In reality, the body *M* proceeds to a point *μ*, which is a vertex of a parallelogram *moμn* constructed by lines *mn* and *mo* and the body *P* proceed to a point *π*, where a prolonged line *Cμ* and the arc *AP* intersect, at the end of the second infinitesimal time *dt*. Consequently, *pi* is equal to *πr*.

By supposing that a point *q* is an intersection of the prolonged line *Cm* and the arc *AP*, *qr* is equal to *2dxdy/y* from the lemmas and because *πr* is equal to *-ddx*, *qπ* is equal to *ddx+2dxdy/*,™. Consequently, *nμ* (*=mo*) is equal to *yddx+2dxdy*. Therefore, since according to him, "...par les principes connus, la force *pi* ou *πr*, multipliée par *Cp* & par la masse *P*, doit faire la même quantité que le produit de la force *mo* par *Cm* & par la masse *M*,"[40] he gives

$$πr×Cp×P=mo×Cm×M...(4-2-1).$$

And from (4-2-1), he obtains

$$yyddx+2ydxdy=-mddx...(4-2-2)$$

and by integration, he gives

$$(vv+m)dx=bdt...(4-2-3).$$

where *bdt* is a integral constant.

Next, because *nμ* is perpendicular to *Cn*, *Cμ* is equal to *y+2dy+ddy=Cn* and from the lemmas, it is equal to *y+2dy+ydx²*. Therefore, he obtains

$$ddy=ydx^2...(4\text{-}2\text{-}4)$$

By solving the equation (4-2-2) for *dx* and substituting the result for (4-2-3), he gives

$$2dyddy = \frac{2bbydydt^2}{(yy+m)^2}.$$

By integration, he obtains

$$dy^2 = -\frac{bbdt^2}{yy+m} + adt^2,$$

where *adt²* is a integral constant.

Eliminating *dt* by using the equation (4-2-4), he obtains the final equation

$$dx = \frac{dy}{\sqrt{(m+yy)}\sqrt{[(yy+m)a/bb-1]}},$$

which coincides with the final result obtained in §.4-1.

§.5 Motions of a rotating tube including a body – D'Alembert's analysis.

D'Alembert poses the problem 2 (Fig.10) in *Traité de Dynamique* as below.

Fig.10

Supposons qu'une verge GA *fixe en* G *& située sur un plan horizontal, soit chargée de deux corps* A, D, *dont l'un* A *soit fixement attaché à la verge, l'autre* D *puisse couler librement le long de la verge par le moyen d'un anneau ; on demande la vitesse de chacun de ces corps à instant, & la courbe décrite par le corps* D.[41]

D'Alembert also proposes the next lemmas to solve the problem (see Fig.11)

Fig.11

LEMME VII.

90. Si deux lignes infiniment petites Pp, Mm, (Fig.23) sont jointes par les lignes finies PM, pm, & qu'on fasse $p\pi=Pp$, & $m\mu=Mm$; je dis 1°. que l'excès de PM sur $\pi\mu$ est égal à deux fois la différence de PM à pm, mois le quarré de l'angle fait entre PM & pm, multiplié par PM.

2°. Que l'angle de $\pi\mu$ avec pm est égal à l'angle de PM avec pm, multiplié par $1+2.(PM-pm)/PM$.[42]

He further proposes a corollary as a special case of this lemma (Fig.12).

Fig.12
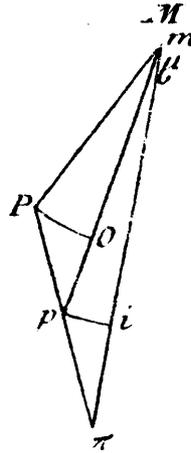
COLLAIRE II.

92. Si les lignes Mm, $m\mu$ (Fig.24) sont $=0$, alors $M\pi-PM=+2pO+PO^2/PM$, & l'angle $pM\pi-PMp=-2pO/PM.PMp$.[43]

Using this corollary, d'Alembert solves the problem. Since Craig C. Fraser explained d'Alembert's method in detail,[44] we will show it briefly.

D'Alembert supposes that during the first infinitesimal instant dt, a body A describes an arc AB and a body D describes a line DE, and that during the second infinitesimal instant dt, if a rod does not disturb the motions of the two bodies, the body A describes an arc BC (=the arc AB) and the body D describes a line Ei (=the line DE). However, since the position of the body D changes with time, the infinitesimal time dt', which the body A describes the arc BC in reality, is different from dt. He designates the distance which the body A describes uniformly with a velocity at the point B during dt' as BQ. Similarly, he signifies the distance, which the body D describes uniformly with a velocity at the point E as Eo. However in reality, since the bodies are restricted each other, at the end of the dt', he supposes that the body A proceeds to the point C and the body D proceeds to the point p.

Considering the motion BQ of the body A during the second instant as composed of the motion BC and CQ, and the motion Eo of the body D during the second instant as composed of the motion Ep and El (=-op and perpendicular to the rod GB), we can realize that the rod will equilibrate when the bodies A and D have the unique motions CQ and El, respectively (d'Alembert's principle). Therefore,

$$A·CQ·GA=D·El·GE...(5-1)$$

By putting GA=a, AB=dx, GD=y, FD=ydy/a, FE=dy and CQ=α, and using the relations $M\pi$-PM=2pO+pO^2/PM and (5-1), we obtain

$$\alpha=2Dydy·dx/(Aa^2+Dy^2)...(5-2).$$

Next, by using the relation $\pi pM\pi-\pi PMp=(-2pO/PM)\pi PMp$, we obtain an equation of a curve described by the body D

$$ddy=dy+ydx^2/a^2+\alpha dy/dx...(5-3).$$

Finally, d'Alembert puts dx=pdy/a to simplify this differential equation and substitutes it for (5-3). Furthermore, by integrating it, he obtains the next equation as a curve described by the body D.

— 39 —

$$dx(=pdy/a)=adyvD/v(Aa^2+Dy^2)[2GD(Aa^2+Dy^2)-1]...(5-4)^{45}$$

Here, G is an integral constant. The equations given by Clairaut (4-2-2) and (4-2-4) correspond to (5-2) and (5-3), respectively.

In the second edition, in Remarque II for the problem II, he discusses motion of a rod and a movable body, where the rod rotates on a vertical plane under the influence of gravity. And he discusses their motion, where the rod rotates with two movable bodies in Remarque IV and with three movable bodies in Remarque V on a horizontal plane. Moreover, he discusses their motion, where it rotates with one fixed body and two movable bodies in Remarque VI and with two fixed bodies and two movable bodies in Remarque VII. We discuss his solution and its relation with Clairaut's solution in detail in the preceding article. In a word, although their metaphysical beliefs are quite different, his method is quite the same as Clairaut [b]'s method from a physical point of view.[46]

§.6 Motions of a rotating tube including a body –Euler's analysis.

Euler discusses motion of a tube including a small body in his memoir E86. Here, we will describe his method briefly for comparison with Johann Bernoulli's and Daniel Bernoulli's methods.

In §.46, a straight tube OC is movable around a fixed point O (see, Fig.13). After a lapse of

Fig.13
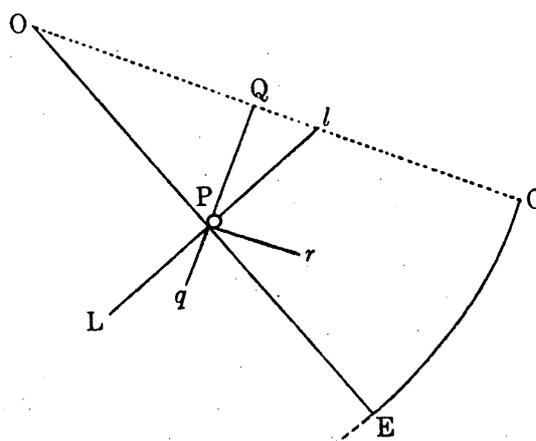
time t, the tube rotates to a position OE and a body in the tube is on a point P. Euler represents a mass of the tube by M, moment of inertia of the tube with respect to the point O by Mkk, a mass of the body by A, πCOE by ω, a length of the tube ($OC=OE$) by f, a distance OP by x and an arc CE by s.

Here, as a moment of force of the tube, Euler gives

$$\frac{2Mkkdd\omega}{dt^2} {}_{47}.$$

Next, he draws a perpendicular PQ to a line OC and puts $OQ=p$, $QP=q$ and he represents a velocity component of the body along the direction Pr by dp/dt, along the direction Pq by dq/dt. Consequently, forces acting on the body along two directions can be expressed by $2Addp/dt^2$, $2Addq/dt^2$, respectively. By decomposing these forces along the directions PE and PL, and expressing p and q by x and ω, he gives force along the direction PE by

$$\frac{2Addx - 2Axd\omega^2}{dt^2}$$

and along the direction PL by

$$\frac{4Adxd\omega + 2Axdd\omega}{dt^2}.$$

In §.48, Euler solves a specific problem. He considers motions of a tube and a small body include in it after any force having acted on a tube only at the first instant. He represents a pressure of the body acting on the tube perpendicularly by P and obtains the next three equations.

$$\frac{2Addx - 2Axd\omega^2}{dt^2} = 0 \,...(6\text{-}1)$$

$$\frac{4Adxd\omega + 2Axdd\omega}{dt^2} = -P \,...(6\text{-}2)$$

$$\frac{2Mkkdd\omega}{dt^2} = Px \,...(6\text{-}3).$$

By substituting (6-3) for (6-2), he gives a pressure P by

$$P = -\frac{4AMkkdxd\omega}{(Mkk + Axx)dt^2}.$$

By eliminating P from (6-2) and (6-3), and integrating it, he obtains $d\omega = \dfrac{Eadt\sqrt{a}}{Axx + Mkk}$, where

$Eadt\sqrt{a}$ is an integral constant. Substituting this equation for (6-1), multiplying $2dx$ and

integrating it, he obtains $dx^2 = -\dfrac{E^2 a^3 dt^2}{A(Axx + Mkk)} + bdt^2$, where bdt^2 is an integral constant. From

this relation, finally he gives

$$dt = \frac{dx\sqrt{A(Axx + Mkk)}}{\sqrt{(bA(Mkk + Axx) - E^2 a^3)}}$$

and

$$d\omega = \frac{Eaadx\sqrt{A}}{\sqrt{(Mkk + Axx)(Ab(Mkk + Axx) - E^2 a^3)}}.$$

Next, Euler proves that in this case, vis viva of the whole system is conserved in §.49 and he considers the motion of the body in the tube and that of the tube when the tube rotates around a fixed point on a vertical plane under the influence of gravity in §.50. In §.51, he considers a case, where M is equal to zero. In §.52-60, he extends §.48 and considers a case where the tube includes three small bodies or further and he adds the condition that M is equal to zero in §.61-63. Finally, he discusses the motion of a curved tube including a small body in it under the influence of no external force in §.68-72.

§.7 Discussions.

We have arranged the characteristics of the resolution methods used by the five scholars in the table. First, we realize similarities between Johann Bernoulli's, Clairaut [b]'s, d'Alembert's and Daniel Bernoulli [a]'s methods, namely similar lemmas or relations. Johann Bernoulli proposes the equation (2-1) (see, Fig.1), Daniel Bernoulli proposes the equation $da = 2dxdy/a$ (see, Fig.3), Clairaut proposes the equation $\pi mCn = dx-2dydx/y$ (see, Fig.8). D'Alembert also proposes the relation $\pi(pM\pi-PMp)=-2pO/PM.PMp$. We can prove that their equations are quite the same as follows.

First, in Johann Bernoulli's equation (2-1), by putting $p^2 + s^2 = y^2$ to use Clairaut's notation and

integrating it, we obtain $ydy=sds$. From §.2, we obtain $d\theta = \dfrac{pds}{p^2 + s^2}$ and according to Clairaut,

we can put $d\theta = dx$. Then we can transform Johann Bernoulli's equation

$$ACB - BCD = \frac{2psds^2}{(pp+ss)^2}$$ to Clairaut's equation $\pi mCn-dx=-2dydx/y$. Second, in Fig.3, we can

express $\pi dAa=da/Ad=2dxdy/a\times 1/Ad$. However, according to Clairaut's notation, we can express Ad as $y+2dy+ddy$. Since the quantity dx/a in Daniel Bernoulli's equation corresponds to dx in Clairaut's equation, by representing $2dxdy/a\times 1/Ad$ by Clairaut's notation and ignoring more than the 3rd order infinitesimal, we can prove that $\pi dAa=2dydx/y= dx-\pi mCn$, namely Daniel Bernoulli's equation coincides with Clairaut's. Needless to say, Clairaut's and d'Alembert's equations are very much the same. This is the case for their second equations, namely $Cn=y+2dy+ydx^2$ (Clairaut) and $M\pi-PM=+2pO+PO^2/PM$ (d'Alembert).

These coincidences are based on the fact that their methods are essentially the same, especially we can realize that d'Alembert's and Clairau's methods are quite the same, by comparing their equations in their lemmas. In the second infinitesimal instant, they regard the motions of the tube and of the body as if they move freely without interaction at the first step, and next, by considering an interaction of the tube and the body and adding this interaction to the free motions, they search for real motions.

Here, so-called d'Alembert's principle is used to divide the motion caused by interaction. Daniel Bernoulli describes generally this principle in a constraint system as below in his article "Demonstrationes theorematum suorum de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae."[48]

> Think that at a given instant the several bodies of the system are freed from one another, and pay no attention to the motion already acquired, since here we speak only of the acceleration or the elementary change of motion. Thus when any body changes its position, the system takes on a configuration different from that it would assume if not freed. Therefore imagine some mechanical cause to restore the system to its proper configuration, and again I seek the change of position arising from this restitution in any body. From both changes you will learn the change of position in the system when not freed, and thence you will obtain the true acceleration or retardation of each body belonging to the system.[49]

In this case, a real motion of the tube (or the body) can be expressed as the sum of a free motion of the tube (or the body) and the constraint motion of the tube (or the body) caused by the body (or the tube). This method corresponds to the principle Clairaut stated at the beginning of §.4-2 (ARTICLE IV. PRICIPE GÉNÉRAL ET DIRECT). D'Alembert also proposes the similar principle.[50] However there is still room to discuss whether their principles are quite the same.

Next, we will speak briefly about Johann Bernoulli's method. Daniel Bernoulli, Clairaut and d'Alembert consider a case, where the tube sets to move by an external force in the first instant and later, the motion of the system is changed only by an interaction between the tube and the body including in it. In addition, Clairaut and d'Alembert consider the motion of the system under the influence of gravity. Euler considers this case, too. On the contrary, Johann Bernoulli solves only a case where the tube rotates uniformly. Consequently, this situation requires that the mass of the body is zero or an external force acts continuously to rotate uniformly and this requirement is quite different from the three researchers'. Moreover, Johann Bernoulli does not require the masses of the tube and the body in his solution and he solves the problem by using only geometrical relation. Then, when he writes that without a restriction of the tube, the body will proceeds to the point D during the second infinitesimal instant, but in reality since the tube proceeds to the point E, the body proceeds to the point E, too, at a first glance we tend to conclude that he uses so-called d'Alembert principle. However, this conclusion is wrong. As stated below, in d'Alembert's principle, a motion of a body, which signifies the product of a mass of the body (m) and the infinitesimal change of a velocity of the body (dv) or the product of m, dv and a distance from an axis of rotation to the body (r), must be considered but Johann Bernoulli never use such a quantity. Consequently, his method is old-fashioned even at that time.

Sometimes Truesdell claims that Daniel Bernoulli's article (Daniel Bernoulli [b)]) is the first to declare the conservation of angular momentum.[51] Therefore, we will examine Clairaut [b)]'s and d'Alembert's methods, which are equivalent to Daniel Bernoulli [a)]'s method. Furthermore, we will examine the relation between Daniel Bernoulli's two methods. According to Truesdell, in 1703,

Jacob Bernoulli is the first to treat the conservation of angular momentum implicitly.[52] In this article, he discusses the center of oscillation of a pendulum. Here, he supposes a quantity, which corresponds to the angular momentum of each mass element, and searches for the length of an equivalent simple pendulum of a composed pendulum by considering the equilibrium of these quantities. Then he invokes the principle of lever in statics on condition that 'the motion itself be regarded as giving rise to forces per unit mass equal to the accelerations reversed.'[53] The quantity considered by him is 'une branch d'un levier'H'une vîtesse' H'un poids ou une puissance' and he regards the products as an extension of the principle of lever in statics, namely the products of the length of a lever and the weight. He declare that

> **Conjecture 1**. *The principle of moment of momentum, as an independent law of mechanics and as a generalization to kinetics of the principle of equilibrium of moments in statics, is due to* James Bernoulli (1686, corrected 1703); *in concept, though not in correct statement, it antedates* Newton's *laws* (1687).[54]

He continues by saying that this principle of conservation is explicitly shown in Daniel Bernoulli [b] and Euler's article.

> ... in 1745 both Daniel Bernoulli and Euler put forward a form of the principle in the course of their solutions of the problem of a mass point constrained to slide within a rigid, rotating tube.[55]

First, we will examine the methods of Clairaut [b], d'Alembert and Daniel Bernoulli [a], which precede that of Daniel Bernoulli [b] and Euler.

Clairaut uses the relation (4-2-1) and it is certain that 'principe connu' signifies the principle of lever. However, a quantity considered by him is the products of the force, the length of a lever and the mass. Since he calls a force 'la impulsion' elsewhere, his quantity is proportional to angular momentum. D'Alembert states that his relation (5-1) signifies equilibrium of the products of the length of a lever, the mass and the motion, here motion means a velocity for him.[56] Therefore, his products also correspond to angular momentum. However, their treatments are only an extension of the principle of equilibrium of lever. Furthermore, Daniel Bernoulli uses the relation ac:dc=m:dD/yy·M in Daniel Bernoulli [a]. Contrary to d'Alembert and Clairaut, since he supposes that the tube has mass, dD/yy·M (une masse cocentrée) corresponds to a mass, which the mass of the tube are concentrated at a point o. And since distances ac and dc mean displacement during an infinitesimal time, namely velocities, ac:dc=m:dD/yy·M means equality of linear momentum of the material point and that of the tube. Therefore, by using the Newton's third law, we can divide the line ac in this way, as shown by the editor of the complete works of Daniel Bernoulli.[57] Since the length of the material point is equal to that of une masse cocentrée, the relation of linear momentum emerges, but not that of angular momentum explicitly. Because in d'Alembert's and Clairaut's resolutions, quantities, which correspond to change in angular momentum during an infinitesimal time in case there are no restriction, vanish and in Daniel Bernoulli's resolution, quantities, which correspond to that of linear momentum, vanish, the integral of this quantity over time is conserved. However, they do not state the products as a concept.

Next, we will examine the relation between Daniel Bernoulli's two articles. Daniel Bernoulli solves the same problem by two methods as shown in §.3-1 and §.3-2. Then for the purpose of proving that these two methods are quite equivalent, we will demonstrate that the two methods give the same curve described by the tube. In §.3-1, he gives a relation $V:u=dx:ds$ for the curve described by the tube. Because this relation is equivalent to another relation $V^2:u^2=dx^2:ds^2$, we will substitute the values of V and u given by Daniel Bernoulli for the latter relation. Because of

$$u^2 = c^2\left\{\frac{ma^2 + MdD}{ma^2} - \frac{MdD}{ma^2}\left(\frac{ma^2 + MdD}{my^2 + MdD}\right)^2\right\} \text{ and } V^2 = c^2\left(\frac{ma^2 + MdD}{my^2 + MdD}\right)^2 \text{, we obtain}$$

$$V^2 : u^2 = ma^2\left(ma^2 + MdD\right) : \left\{\left(my^2 + MdD\right)^2 - MdD\left(ma^2 + MdD\right)\right\}.$$

From $V^2:u^2=dx^2:ds^2$, we obtain

$$dx^2 : ds^2 = ma^2\left(ma^2 + MdD\right) : \left\{\left(my^2 + MdD\right)^2 - MdD\left(ma^2 + MdD\right)\right\}.$$

Substituting $ds^2 = \left(\dfrac{y}{a}dx\right)^2 + dy^2$ for this proportion and solving it for dx, we obtain

$$dx = \frac{a\sqrt{MdD + ma^2}}{\sqrt{my^4 + \left(MdD - ma^2\right)y^2 - a^2 \cdot MdD}} \cdot dy \, .$$

By comparing this equation with $dy = a^2 dx \times \dfrac{\sqrt{MdD + ma^2}}{\sqrt{\left(MdDa^2 + ma^2x^2\right)\left(x^2 - a^2\right)}}$, we can conclude

that both equations are perfectly the same. And in the equation

$dy = a^2 dx \times \dfrac{\sqrt{MdD + s\delta\Delta}}{\sqrt{\left(MdDa^2 + s\delta\Delta x^2\right)\left(x^2 - a^2\right)}}$, which Daniel Bernoulli gives in §.3-2, when the

system has only one body, we obtain the latest equation. Finally, we can conclude that the both methods are perfectly the same.

Then, what is the difference between two methods? In his letter to Euler, Daniel Bernoulli writes as below.

> Ich habe auch über dergleichen problemata einige compendia, sonderlich ratione vis acceleratricis globi in tubo, vermittelst welcher ich kann die differentialia 2 $^{\text{di}}$ gradus evitiren und die ganze Solution kürzer machen, die Zeit erlaubt mir aber nicht, solche nunmehro zu explieiren.[58]

And in another letter to Euler, he writes as below.

> Ew. Principium conservationis momentorum motus rotatorii abbreviirt freylich die problemata de motu corporis in tubo : ich hatte aber solches auch schon observirt, und ist ein corollarium von der methodo directa, die ich Ihnen einmal für einen gewissen casum überschrieben hatte.[59]

Therefore, according to him, his latter solution has its novelty in respect that he does not use the second order differential but conservation of le momentum du mouvement circulatoire.

First, we consider conservation of le momentum du mouvement circulatoire. He uses the quantity $dD/yyHM$, which we took up in §.3-2, in both methods. As described above, he derives this quantity in his preceding article "De Variatione motuum a Percussione excentrica." According to the article, a plane ABC on a horizontal plane is rotatable around a point D (Fig.14). A force (potentia)

Fig.14
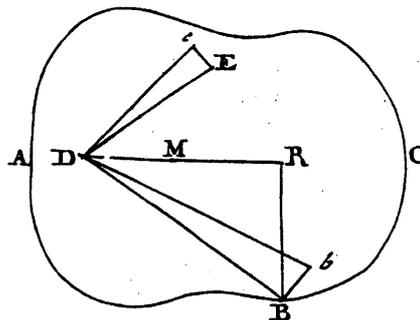
acts on a point B, whose infinitesimal mass is μ, and the plane ABC rotates around the point D with acceleration. He searches for a relation between μ and m, which is an infinitesimal mass of a point E on the plane ABC, when the point E has the same acceleration (acceleratio) as the point B. The result is $\mu{:}m = DE^2{:}DB^2$.[60] Here, we interpret according to the editor of complete works of Daniel Bernoulli, that he searches for a relation in which the point B and the point E have the same angular velocity (ω). Then, by putting the increase of the point B during an infinitesimal time dt $(dv_B) = DB \cdot d\omega$, and that of the point E $(dv_E) = DE \cdot d\omega$, a quantity of the point B, $\mu \cdot DB \cdot dv_B = \mu \cdot DB^2 \cdot d\omega$ is equal

to a quantity of the point E, $m \cdot DE \cdot dv_E = m \cdot DE^2 \cdot d\omega$. This relation means that the change of angular momentum of the point B with respect to the point D is equal to that of the point E with respect to the point D. By expansion, regarding the change of angular momentum of the point B with respect to the point D as equal to the change of angular momentum of the whole plane ABC with respect to the point D, we obtain $\mu \cdot DB^2 \cdot d\omega = I \cdot d\omega$. Here, I signifies momentum of inertia of the plane ABC with respect to the point D. Therefore, we obtain $\mu = I/DB^2$, which corresponds to $dD/yy \times M$ described above.

Then, he states le momentum du mouvement circulatoire, which corresponds to conservation of angular momentum, in his "Nouveau Probleme de Mécanique" for the first time but already its idea can be recognized in his "De Variatione motuum a Percussione excentrica." Therefore, we can understand that his solution given by his letter includes the concept of conservation of angular momentum implicitly.

In "Nouveau Probleme de Mécanique", Daniel Bernoulli proves that le momentum du mouvement circulatoire of the tube and the body is conserved respectively when they move freely without constraint. Next, he proves that when considering an interaction, le momentum du mouvement circulatoire of the whole system is invariable. For this demonstration, instead of the tube whose mass distributes wholly, he considers an interaction of the body in the tube and the tube whose mass, $dD/yyHM$, concentrates in the point o (Fig.5). And he divides the line ad into

$$ac : dc = m : \frac{dD}{yy}M$$. Consequently, multiplying the impulse by a distance from the center of

rotation A to a point which an interaction acts, the increase of le momentum du mouvement circulatoire of one is naturally equal to the decrease of le momentum du mouvement circulatoire. As for le momentum du mouvement circulatoire, he states that "c'est à l'imitation de ce qu'on appelle *momentum* d'une force qui agit sur un levier."[61] It is that his concept of le momentum du mouvement circulatoire is an extension of the principle of lever.

Finally we can conclude that the difference between Daniel Bernoulli's two methods is that he does not use the second difference in his "Nouveau Probleme de Mécanique." Namely, his method stated in his letter uses the second difference and geometrical relations. The same thing can be said of Johann Bernoulli's, d'Alembert's and Clairaut [b]'s methods in §.4-2. These methods need a special technique in a specific problem and it is difficult to set equations of motion generally in any situation. On the contrary, the method in "Nouveau Probleme de Mécanique" does not need such a special technique and we can obtain equations of motion easily, just like Clairaut [a]'s method. However, there exists the essential difference between giving a name as a notion and using unconsciously, though Clairaut makes an excuse![62]

This principle of conservation is pointed out by Euler and he further generalizes this principle. Truesdell states that

The method used by Euler is the same as James Bernoulli's, but it is clearly explained, and Euler gives (5) explicitly in the special case of plane motion about a fixed axis ($L=Ia$); his alleged proof is no more than an assertion of the extended law of the lever.

Here, the equation (5) signifies $dH/dt=L$ and H is angular momentum (or 'the moment of momentum') and L is 'the total torque exerted by the external forces.'

When a body rotating around a point O changes its motion by an external force (Fig.15), Euler

Fig.15
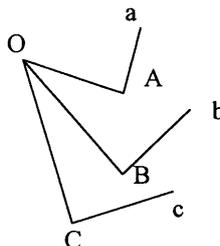
searches its motion by equating the sum of moment of force gained by each element of the body to moment of force given by an external force to the body. He searches for a force that acts on a point A, whose direction Aa is perpendicular to line OA and then multiplies this force by distance OA and sums up the products over the body. He names the final quantity the sum of moment of force (omnium harum virium summa momentorum ad axem). Then, by equating this sum to moment of force caused by an external force, he derives the equation of a rotating body.[63] It is certain that his starting point is the principle of lever, because he considers moment of force, which is equal to the products of the length of a lever and force. However, the establishment of the principle of conservation of angular momentum from the principle of lever in statics is not simple. In discussing the history for searching the length of an equivalent simple pendulum of a composed pendulum, Christiane Vilain makes clear the process of development from Jacob Bernoulli's quantity, which is the products of the length of a lever, the velocity and the weight derived from the principle of lever in statics, via Johann Bernoulli's and Hermann's quantity, to Euler's quantity, which is moment of force with no relation to a lever.[64] Therefore, Truesdell's evaluation to Euler is too simple.

Furthermore, Euler gives a general equation for a rotating body in the two dimensions $2Mkkdd\omega/dt^2=Px$ (which corresponds to $dH/dt=L$). And Euler, contrary to Daniel Bernoulli, uses a concept of moment of inertia (momentum inertiae corporis=Mkk). Furthermore, he demonstrates that in this special case, namely the motion of a rotating tube including a material point, angular momentum (momentum motus gyratorii) of the system is coserved.[65]

From another point of view, we can divide their methods into two groups; one includes Euler and d'Alembert, the other includes Daniel Bernoulli, Clairaut. Euler declares at the beginning of his article treating a body in a rotating tube that:

> Quod negotium cum suscepissem, praecipue in hoc elaboravi, ut huiusmodi problematum solutiones ex primis mechanicae principiis investigarem, neque ad has principiis inde demum derivatis, cuiusmodi est conservatio virium virarum, uterer ; non quasi huius principii veritatem in dubium vocarem, sed potius ut consensu mearum solutionum cum iis, quae ex principio isto sint deductae, eius veritas etiam illis, qui adhuc de eo dubitant, plenissime confirmaretur. Tum vero saepe occurrunt casus, quibus hoc principium alias utilissimum, omni usu ad solutionem obtinendam caret, eum contra prima mechanicae principia, si recte adhibeantur, semper ad solutionem perducere debeant.[66]

Namely, he uses the primary principles, not conservation of vis viva. He also writes in another article that:

> ...l'une [méthode], tirée des primiers principes de la Mécanique, par le moyen desquels on trouve, à chaque instant, le changement tant dans la vitesse que dans la direction, causé par les forces sollicitantes....Mais la première méthode, quoiqu'elle soit beaucoup plus difficile que l'autre [fondée sur des principes dérivatifs],. Semble pourtant être plus naturelle, parce qu'elle montre à chaque instant, non seulement l'état du mouvement, mais encore les véritables causes de tous les changements qui arrivent successivement.[67]

Next we will states his primary principles used in §.6 of this memoir briefly. In Fig.11, he sets $OQ=p$, $QP=q$ and signifies a force acting on the body P along the direction OQ as F_{OQ}, a force acting along the direction QP as F_{QP} and firstly he gives the equations of motion for the body P by

$$F_{OQ}=2A \cdot ddp/dt^2, \ F_{QP}=2A \cdot ddq/dt^2$$

and by signifying moment of inertia with respect to the point O as Mkk, a rotational angle around the point O as ω and a moment of force acting on the tube as N, secondly he gives the equation of rotation of the tube by

$$N=2Mkk \cdot dd\omega/dt^2.$$

It is these equations that are used at present time to solve such a problem

D'Alembert writes as below.

> L'élégance dans la solution d'un Problême, consistant surtout à n'y employer que des principes directs & en très-petit nombre, on ne sera pas surpris que l'uniformité qui regne dans toutes mes solutions, & que j'ai eue principalement en vûe, les rende quelquefois un

peu plus longues, que si je les avois déduites de principes moins directs.[68]

For Euler and d'Alembert, mechanics must be deduced from primary principles, though their system are quite different. Clairaut, on the other hand, uses various methods to solve the problem. And as for Daniel Bernoulli, the editor of the complete works of Daniel Bernoulli writes that

> While Daniel Bernoulli used this principle for solving various problems, d'Alembert stated it as *the* universal principle of dynamics able to replace even Newton's notion of force and his laws of motion. ...here it suffices to say that Bernoulli never made such a claim.[69]

Finally, we can place their methods as follows. Johann Bernoulli's method treats a special case of a rotating tube including a body in it geometrically. We can not find a mass and motion of the body in it. Next, Daniel Bernoulli [a]'s method, Clairaut [b]'s method and d'Alembert's method use the second differential and geometrical relations, which are based on the 17th century tradition. In addition, the concept of une masse concetrée introduced by Daniel Bernoulli, is his device to treat an extended body on the basis of the tradition of Huygens' origin. Daniel Bernoulli [b]'s method , which is based on this tradition derived from the 17th century, introduces conservation of angular momentum and is new one. And Clairaut [a]'s method is composed by a skillful combination of conservation of vis viva and $\varphi dt = dv$ in the 18th century fashion. However, their methods are proposed to solve given problems skillfully under the specific conditions and are not the primary principles applicable to all cases. On the contrary, the method given by Euler in §.6 treats not only the special case, namely motion of a tube including a material point but also by introducing a concept of moment of inertia, a general method to solve motion of rigid body.

notes

1 Domenico Bertoloni Meli, "The emergence of reference frames and the transformation of mechanics in the Enlightenment", *Historical Studies in the Physical and Biological Science*, vol.23, 1993, p.306.

2 Euler, *Opera Omnia*, ser.4A, vol. 2, Einleitung Euler-Johann I Bernoulli, p.59. Unfortunately, this letter was missing.

3 Euler, *Opera Omnia*, ser.4A, vol.2, Nr.33, p.412.

4 See, Johann Bernoulli's letters to Euler dated on March 15, on August 27 and on March, 1742. (Euler, *Opera Omnia*, ser.4A, vol.2, Nr.33, 34, 35.)

5 Euler, *Opera Omnia*, ser.4A, vol.2, Einleitung Euler-Johann I Bernoulli, p.59.

6 Euler, *Opera Omnia*, ser.4A, vol.2, Nr.33, p.413.

7 Euler, *Opera Omnia*, ser.4A, vol.2, Nr.34, pp.424-427.

8 Johannis Bernoulli, *Opera Omnia*, vol.4, N° CLXXVI, "De curva quam describit corpus inclusum in tubo circulante", pp.248-252, (JB176).

9 "Die hier in französischer Sprache angeführte Darlegung der Lösung wurde von J. Bernoulli später etwas umgearbeitet und als selbständiger Artikel im 4. Band seiner *Opera* lateinisch publiziert(JB. 176)." (Euler, *Opera Omnia*, ser.4A, vol.2, Nr.34, p.435, note 12.)

10 *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIeme siècle*, ed. P.H. Fuss, 2 vols. (St. Petersburg, 1843), Vol. II, Lettre de Daniel Bernoulli à Léonard Euler, Lettres XXII, XXIII, XXV, XXVIXXVII, XXIX, XXX, XXXI, XXXII and XL.

11 Fuss, ibid. Lettre XXVI.

12 Daniel Bernoulli, "Nouveau probleme de mécanique résolu par Mr. Daniel Bernoulli," *Mémoires de l'Académie des Sciences de Berlin*, 1745 (1746), pp.54-70 ; *Die Werke von Daniel Bernoulli*, ed. D. Speiser, (Birkhäuser Verlag, 1987), Band 3, Mechanik, pp.179-196.

13 Euler, *Opera Omnia*, ser.4A, vol.5, n°18, 19 and 20.

14 Clairaut, "Sur quelques principes qui donnent la solution d'un grand nombre de problèmes de dynamique", Mémoires de l'Académie des Sciences de Paris, (1742), 1745, pp.1-52.

15 Euler, "De motu corporum in superficiebus mobilibus", Opuscula varii argumenti 1 : *Opera Omnia* ser.2, vol. 6, pp.75-174, (E86).

16 D'Alembert, *Traité de Dynamique*, (Paris, 1er ed. 1743 ; 2nd ed. 1758), Problême II.

17 Here, we use a figure in Latin version, because the French version does not have a correct figure.

18 Exactly speaking, *EC* is equal to $x+2dx+ddx$, but by approximating πDCE within 2nd order

infinitesimal, we can reach at this equation.

[19] "*n*" means a natural logarithms "*ln*".

[20] In the Latin version, he deduces the relation $2dydx^3 = ds^3 \cdot d(xdy/ds)$ from $\pi DCE = \pi BCA - \pi DCB$. Then, by substituting this relation for $dy = xdz/a$ and considering that dz is constant, he

obtains $2xdx^3 = ds^3 \left(\dfrac{x^2}{ds} \right)$, namely $2dx \cdot ds^2 - xds \cdot dds = 2dx^3$. Next he substitutes this result

for two equations $ds^2 = dx^2 + dy^2 = dx^2 + \dfrac{x^2 dz^2}{a^2}$ and $ds \cdot dds = dx \cdot ddx + \dfrac{xdx \cdot dz^2}{a^2}$. Then

he deduces $xdx \cdot dz^2 - b^2 dz^2 = a^2 dx^2$ or $dz = \dfrac{adx}{\sqrt{x^2 - b^2}}$, where b is constant. By integrating

this relation, he finally obtains $z = \displaystyle\int \dfrac{adx}{\sqrt{x^2 - b^2}} = a\log\left(x + \sqrt{x^2 - b^2}\right)$.

[21] Fuss, ibid., lettre XXVI.

[22] Fuss, ibid., lettre XXVI, p.500.

[23] For derivation of this relation, see §6.

[24] "Num vero concipienda est potentia, quae globum versus tubum premat, et altera aequalis, quae tubo in *a* applicata, eundem premat versus globum; hoc modo globus et tubus convenient in *c*; erit itaque position globi post secundum tempusculum *dt* in *c*, et positio tubi in *Acf*;..."(Fuss, ibid., lettre XXVI, p.501.)

[25] For derivation of this relation, see §6.

[26] We can demonstrate this relation as below. In Fig.16, because we can regard that a line *og* is

Fig.16
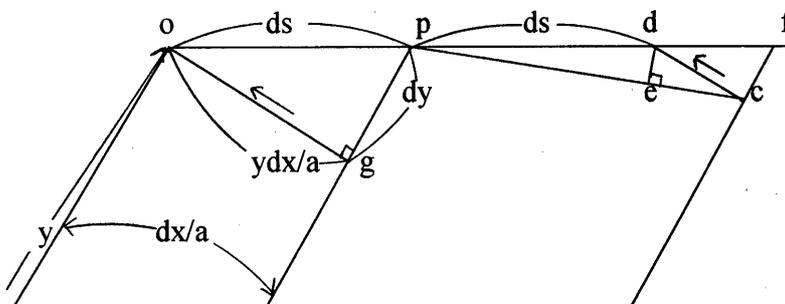
parallel to a line *cd*, πpog is equal to πfdc. Moreover, because a line *de* is perpendicular to a line *ec* and we can regards that a line *pd* is perpendicular to the line *de*, we obtain $\pi pog = \pi fdc$. Therefore, triangle *ogp* is similar to triangle *ced*. Then, we obtain *og* : *op*=*ce* : *cd* namely *ydx/a* : *ds*=*ce* : *dc*. Finally, we have a conculsion, *ec*=*cd*≅(*ydx/ads*)=[*MdD/(myy+MdD)*]≅(*2dx*≅*dy/a*)≅(*ydx/ads*).

[27] We can derive this relation as follows. A pressure of the sphere caused by the tube is equal to a perpendicular component to the tube *Af* of $m \times dds/dt^2$, namely a component along the direction *dc* (see, Fig.3). Because of the relation $ydx/a : ds = ce : dc$, a required pressure is $m \times dds/dt^2 \times ds/(ydx/a)$. By considering $dt = dx/V$, we obtain

$\dfrac{2VV}{a} \cdot \dfrac{dy}{dx} \cdot \dfrac{MdD}{myy + MdD} \cdot m$. Finally, by considering a factor 2 caused by Euler's unit, we obtain

$\dfrac{4VV}{a} \cdot \dfrac{dy}{dx} \cdot \dfrac{MdD}{myy + MdD} \cdot m$ as a pressure.

[28] As for this relation, see, §6.

[29] Daniel Bernoulli, *Werke* 3, General Introduction, pXXVI.

[30] We must pay attention to the notations x and y, because their meanings are interchanged from §.XVI.

[31] Daniel Bernoulli derives this relation as follows. He obtains $x'dp = xdp'$ from $dp = \dfrac{xV^2}{a^2}dt$

and $dp' = \dfrac{x'V^2}{a^2}dt$. Because dx and dxN are proportional to p and pN, respectively, he obtains

$pdx' = p'dx$. By adding two equations, he gives $x'dp + pdx' = xdp' + p'dx$. And by integrating this equation, he obtains $x'p = xp'$. He derives $dx'/x' = dx/x$ from $x'p = xp'$ and

$pdx' = p'dx$. Finally by integrating the latest result, he derives $x'/a' = x/a$, here a integral constant is determined adequately.

[32] It is the case that is solved by Johann Bernoulli.

[33] Clairaut, ibid.

[34] We can derive this equation as follows. By representing la force accélératrice by φ, a velocity of a body by u and a distance described by the body by e, we obtain $\varphi dt = du$ and then, $\varphi de = udu$ (See, for example, J.L.d'Alembert, *Traite de Dynamique*, (2nd ed.), p.24). A force acting on the body along the direction CM is a centrifugal force and its quantity is equal to a velocity component at a point M along the direction MR to the 2nd power over CM, namely $y\,dx^2/dt^2$, which is equal to φ. By substituting $de=dy$, $u=dy/dt$ and $du=d(dy/dt)=ddy/dt$ for $\varphi de=udu$, we obtain the required equation.

[35] Here, a stands for constant. The first term ,the second term and the third term of the left side are quantities derive from a velocity of the body M along the direction MR, along the direction Rm and that of the body P, respectively.

[36] Clairaut, ibid., pp.21-22.

[37] Meli, ibid., p.307.

[38] Clairaut, ibid., pp.24.

[39] Clairaut, ibid., pp.25.

[40] Clairaut, ibid., p.25.

[41] D'Alembert, ibid., p.104.

[42] D'Alembert, ibid., p.102.

[43] D'Alembert, ibid., p.104.

[44] Craig G. Fraser, "D'Alembert's principle: the original formation and application in Jean d'Alembert's *Traité de Dynamique*," *Centaurus 28. Copenhagen, 1985*, pp.31-61(Part One) & pp.145-159(Part Two).

[45] Fraser derives the equations (5-1), (5-2) and (5-3) in his article in detail.

[46] Apparently Clairaut solves the problem under the assumption that dt is constant and d'Alembert supposes that dt is variable. However, their methods are quite the same except for this condition.

[47] According to the modern notation, this quantity corresponds to $I \times \dfrac{d\omega}{dt}$, where I signifies a moment of inertia of the tube with respect to the point O and ω signifies an angular velocity of the tube.

[48] *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, vol. XII, pp.162-173, 1734/1735, (1740).

[49] Translated by C. Truesdell (C. Truesdell, *The Rational Mechanics*, in *Leonhardi Euleri Opera Omnia*, ser.2, vol.10, pp.159-160.). The original sentences are :
"Puta in systemate ad momentum temporis corpora singula a se inuicem solui, nulla facta attentione ad motum iam acquisitum, quia hic de acceleratione seu mutatione motus elementari tantum sermo est : ita quolibet corpore situm suum mutante, systema aliam accepit figuram, quam non-solutum habere debebat : Igitur finge causam mechanicam quamcunque systema in debitam figuram

restituentem atque, rursus inquiro in mutationem situs ab hac restitutione oram in quolibet corpore ; et ex vtraque mutatione intelliges mutationem situs in systemate non soluto, indeque accelerationem retardationemue veram cuiusuis corporis ad systema pertinentis obtinebis."

[50] D'Alembert, T.D., Problême general, pp.73-75.

[51] C. Truesdell, *Essay in the History of Mechanics*, (Springer-Verkag Berlin Heidelberg New York 1968), Whence the Law of Moment of Momentum?, pp.252-256.

[52] Jacob Bernoulli, "Démonstration generale du centre de balancement ou d'oscillation, tirée de la nature du levier," *Mém. Acad. Sci. Paris*, 1703, pp.78-84.

[53] C. Truesdell, ibid., III. Reactions of Late Baroque Mechanics to Success, Conjecture, Error, and Failure in Newton's Principia, p.156.

[54] C. Truesdell, ibid., p.252.

[55] C. Truesdell, ibid., p.252-254.

[56] D'Alembert, *T.D.*, p.73.

[57] Daniel Bernoulli, *Werke* 3, introduction, p.93.

[58] Fuss, ibid., lettre XXVII, p.511.

[59] Fuss, ibid., lettre XXIX, pp.525-526.

[60] His derivation of this proportion is hard to understand for us. For example, the editor of complete works of Daniel Bernoulli writes as follows. "What Bernoulli writes is difficult to follow in detail" (Daniel Bernoulli, *Werke* 3, Introduction, p.43.)

[61] Daniel Bernoulli, *Werke* 3, p.180.

[62] "Au reste j'ai été charmé de votre Theorême de la conservation des momens rotatoires, mais ce qui m'a poqué c'est que si j'avois fait un peu de reflexion sur mes equations, je l'aurois trouvé aussi. Car dans toutes les questions de même nature que j'ai donné à l'Academie, je suis arrivé deux equations finales dont l'une est celle qu'on peut trouver par la conservation des forces vives et dont l'autre est la même que celle que donne votre conservation des momens rotatoires." (Euler, *Opera Omnia*, ser.4A, vol.6, *Correspondance de Leonhard Euler avec A. C. Clairaut, J. d'Alembert et J. L. Lagrange*, Clairaut à Euler (No. 19), p.146.)

[63] He derives an equation of a rotating body as below.
He puts the rotating velocity of a point in the body ou, whose distance from the axis is f. He supposes that the point increases its velocity du during an infinitesimal rotation dω. A point A has a velocity OA·ou/f and this velocity is produced by free fall from the height OA^2.u/ff. During an infinitesimal rotation dω, the point A proceeds OA·dω, and gains an infinitesimal increase in height OA^2.du/ff. Therefore, the particle is acted by a force along the direction Aa,
$$OA^2.du/ff :OA.df\ddot{O}=A.OA.du/ffd\omega$$
Similarly, points B and C receive forces B·OB·du/ffdω and C·OC·du/ffdω, respectively. Therefore, the sum of moment of force with respect to the axis O is
$$du/ffd\omega \cdot (A.OA^2+B.OB^2+C.OC^2+...).$$
By designating external moment of force as Pg, he gets the equation for the acceleration of a rotating body,
$$du=(Pg.ffd\omega)/ (A.OA^2+B.OB^2+C.OC^2+...).$$
By putting $(A.OA^2+B.OB^2+C.OC^2+...)$=Mkk (moment of inertia), he gives
$$du=(Pg.ffd\omega)/Mkk...(1).$$
By using the relation $du=2ff.d\omega.dd\omega/dt^2$, which derives from $u=ffd\omega^2/dt^2$, he obtains $Mkk.du/ffd\omega=2Mkkdd\omega/dt^2$. Therefore, he rewrites (1) into another form
$$2Mkkdd\omega/dt^2=Px...(2).$$

[64] Christiane Vilain, "La question du "Centre d'oscillation" de 1703 à 1743," to be published in *La Revue Physis*.

[65] Euler, ibid., §.54, p.126.

[66] Euler, E86, p.75.

[67] Euler, "Dissertation sur le Mouvement des Corps enfermes dans un Tube droit mobile autour d'un Axe fixe", *Opera Omnia*, ser.2, vol.7, p.267.

[68] D'Alembert, *T.D.*, Discours Préliminaire, p.xxxiij.

[69] Daniel Bernoulli, *Werke* 3, General Introduction, p.XXVI.

Analysis of motion of a rotating tube including a material point by Johann Bernoulli, Daniel Bernoulli, Clairaut, d'Alembert and Euler

	Solution	Mass of the tube (rod)	Angular momentum	Primary principle	Note
Joh. Bernoulli	Geometrical relation	×		×	
Clairaut [a)]	f dt=dv & Conservation of vis viva	×		×	
Clairaut [b)]	Geometrical relation & d'Alembert's principle	×	Length of the lever × force × mass	×	Equivalent
D'Alembert	Geometrical relation & d'Alembert's principle	×	Length of the lever × mass × motion[(1)]	○	Equivalent
D. Bernoulli [a)]	Geometrical relation & d'Alembert's principle	○ (une masse concentrée)	ac :dc=m :dD/yy ·M	×	Equivalent
D. Bernoulli [b)]	Conservation of angular momentum	○ (une masse concentrée)	m.m.c.[(2)]=conservation of mvr (imitation of momentum of force)	×	
Euler	F=ma & N=I·dω/dt	○(moment of inertia)	$Px=2Mkk \cdot dd\omega/dt^{2}$ [(3)]	○	

(1) motion=velocity of a mass(T.D.p.73).
(2) momentum du mouvement circulatoire.
(3) Left side…"omnium harum virium summa momentorum ad axem";   Mkk…"momentum inertiae corporis".
*. D'Arcy also solves the same problem by using the conservation of angular momentum (the conservation of areal velocity) and the conservation of vis viva (1746).