Analysis of motion of a rotating tube including a material point by Johann Bernoulli, Daniel Bernoulli, Clairaut, d’Alembert and Euler

Ryoichi NAKATA

§1. Introduction.

One of the most attractive problems in 1740s was a motion of a rotating tube around a fixed point and that of a small body included in it. Representative mechanicians in those days, such as Daniel Bernoulli, Alexis-Claude Clairaut, Leonhard Euler and Jean le Rond d’Alembert, tried to solve this problem.1 This problem was later applied to astronomical mechanics. It was posed by Euler for Johann Bernoulli in his letter in 1741.2 According to Johann Bernoulli’s letter to Euler dated on March 15, 1742, this problem is to determine a motion of a rotating tube around a fixed point and that of a body included in it under the influence of the gravity.

Sit tubus seu canalis (sive gravis sive gravitatis exprs) mobilis circa axem fixum, in quo versetur globus, qui ob gravitatem in tubo sine frictione descendat (et quidem, quod sine dubio subintelligis, non rotando sed fluendo) simulque tubo motum inducat: quovis tempore determinare situm tubi et globi in tubo, itemque utriusque celeritatem.3

Johann Bernoulli tried to solve this problem in his succeeding letter but he failed to solve it.4

Johann Bernoulli in reality solved a problem posed by J.S. König in autumn of 1743 when König studied with Clairaut and Maupertuis.5 It is as follows,

Determiner la courbe, que decrit un corps renfermé dans un Tuyau pendant que le tuyau se meut uniformement autour d’un Centre sur un plan horizontal.6

According to Johann Bernoulli’s letter to Euler dated on August 27, 1742, these two problems seem to be similar but for him, they were quite different. He could solve the problem posed by König but the problem posed by Euler was unsolvable for him. He admitted that there was a great disparity between them and summarized difference as follows. 1. König considers a case where the gravity does not exist (a motion on a horizontal plane) but Euler poses motion under the influence of gravity. 2. A tube rotates uniformly in the former case but it does not rotate uniformly in the latter case. 3. Motion of a body included in a tube depends on the tube in the former case but in the latter case, a tube is weightless and its motion depends on a heavy body.

Johann Bernoulli published the resolution of this problem twice. First, he wrote it in French as a supplement in his letter to Euler dated on August 27, 17427 and after that, he rewrote and published it in the 4th volume of his complete works in Latin.9

Euler also posed this problem to Daniel Bernoulli and Clairaut. Daniel Bernoulli frequently mentioned it in a series of letters to Euler.10 Especially, in his letter dated on October 20, 1742,11 he discussed a motion of a heavy straight tube rotating around a fixed point on a horizontal plane and that of a material point included in it. Next, he solved this problem by using the principle of conservation of “le momentum du mouvement circulatoire”, which corresponds to the conservation of angular momentum for us, and published it in his memoir.12

Clairaut also gave an outline of this resolution in his letter to Euler13 and published it in his memoir.14 Euler himself published his resolution in his memoir.15 D’Alembert alone discussed this problem in his book independently.16

In this article, we will examine Johann Bernoulli’s and Daniel Bernoulli’s methods to solve this problem in §2 and 3. In §. 4, 5 and 6, we will summarize Clairaut’s, d’Alembert’s and Euler’s methods. Finally, we will compare and examine their five methods and discuss their positions in the 18th century mechanics, which was on the way to constitute the modern physics.

§2 Motions of a rotating tube including a body – Johann Bernoulli’s analysis.

Johann Bernoulli’s two methods are essentially the same. Then, we will employ the French version here.

First, he introduces a lemma (Fig.1).
**Lemme.** Si d'un point $C$ l'on tire les droites, $CA$, $CB$, $CD$, qui coupent sur $PD$ les parties $AB$, $BD$, in finiment petites et égales, la différence des angles $ACB$ et $BCD$ c'est à dire

$$ACB - BCD = \frac{(2pds)^2}{(pp+ss)^2}$$

en nommant la constante $CP=p$, perpendiculaire sur la droite variable $PA=s$. Cela se démontre facilement en différentiant l'angle $ACB$ dans la supposition de $ds$ constant.

We can demonstrate the equation (2-1) as below. By putting $\pi PCA = \theta$, we obtain $\pi ACB = d\theta$ and by differentiating an equation $\cos \theta = p/\sqrt{p^2+s^2}$ we obtain $-\sin \theta \cdot d\theta = -1/2 \cdot (p^2+s^2)^{-3/2} \cdot p \cdot 2ds$. Substituting $\sin \theta = s/\sqrt{p^2+s^2}$ for this equation and solving it for $d\theta$, the resolution is $d\theta = \frac{pds}{p^2 + s^2}$.

Then, by differentiating it once more under the condition of $ds$ constant, we can obtain

$$|\angle ACB - \angle BCD| = \frac{2pds^2}{(p^2 + s^2)^2}.$$ 

Next, he solves a problem.

**Probleme.** Determine la courbe que décrit un corps renfermé dans un tuyau pendent que le tuyau se met uniformément autour d'un centre sur un plan horizontal.

In Fig.2, $ABE$ represents a curve described by a body when a tube $CA$ rotates around a point $C$. We put $CA = x$, $AB = ds$ and $BH = dy$. The body describes an infinitesimal line $AB$ during an infinitesimal time. If it were not restricted, it would describe another infinitesimal line $BD = AB$, which is a prolonged line of $AB$ during a succeeding equal infinitesimal time. But the tube proceeds to $CE$ at
the end of the second infinitesimal time and $\pi ECB$ is equal to $\pi BCA$. Then, in reality the body proceeds to a point $E$. Here, the tube rotates equal angles during two equal infinitesimal durations.

He drops perpendiculars $CP$ and $CR$ to lines $BP$ and $ER$, respectively and since $\pi CAG$ is equal to $\pi CGA$ and $\pi R$, because of $CA = CG$, he obtains the relation that triangle $CPA$ is similar to triangle $AGB$. Then he can obtain $p = CP = x \cos \angle ACP = x \cos \angle BAG = x \frac{dy}{ds}$. Similarly he obtains the relation $PA = x \frac{dx}{ds}$, therefore, $|\angle ACB - \angle BCD| = 2\frac{dydx}{xx}$. Furthermore, he obtains

$$\frac{d(xdy/\overline{ds})}{xdy/\overline{dx}} = \frac{DF}{ds},$$

because of $RQ/QB = DF/BF$ and from this,

$$DF = \frac{ds^2 \cdot d(xdy/\overline{ds})}{xdy/\overline{dx}}.$$

Next, he obtains

$$dx \cdot ds :: \frac{ds^2 \cdot d(xdy/\overline{ds})}{xdy/\overline{dx}} \cdot DE,$$

because triangle $HDB$ is similar to triangle $FDE$. By considering that $ds$ is constant, he obtains

$$DE = \frac{ds^2 \cdot d(xdy)}{xdy^2}.$$

Therefore, he gives the relation

$$\frac{DE}{EC} = \frac{ds^2 \cdot d(xdy)}{xdy^2} = \angle DCE. \quad \text{18}$$

From the relation $\text{e}DCE = \text{e}BCA - \text{e}DCB$, he obtains

$$x \frac{dy^2}{dy} - x \frac{dy^2}{dy} + 2x \frac{dy^2}{dy} = dy \frac{dx}{dx} = dx \frac{(x^2 - dy^2)}{dx - dy^2},$$

he obtains

$$\frac{dy}{dy} + \frac{2dydy}{dx - dy^2} = \frac{dx}{x}.$$  

By considering $dx^2 - dy^2 = ds^2 - 2dy^2$, this relation can be transformed into

$$\frac{dy}{dy} + \frac{2dydy}{ds^2 - 2dy^2} = \frac{dx}{x}.$$  

By integrating it, he obtains

$$l(n\overline{dy}) - 1/2 l(ds^2 - 2dy^2) = lx, \quad \text{19}$$

where $n$ is a constant.

From this relation, he gives

$$\frac{ndy}{\sqrt{dx^2 - dy^2}} = x$$

or

$$dy = \frac{xdx}{\sqrt{mx + xx}}.$$

By replacing $dy$ with $a$ and $dz$, which are equal to a radius $CM$ and an infinitesimal element of an arc $MN$, respectively, he obtains the relation
\[ dz = \frac{adx}{\sqrt{mn + xx}}. \]

By integration, he reaches the final relation
\[ z = \int \left( x + \sqrt{mn + xx} \right). \]

This equation represents a curve described by the body in the tube. 20

\section*{§.3 Motions of a rotating tube including a body -- Daniel Bernoulli’s analysis.}

\section*{§.3-1 Analysis in the letter to Euler in 1742 (Daniel Bernoulli 21).}

Daniel Bernoulli, in his letter to Euler dated on December 20, 1742, wrote that “Ew. Problema generalissimum circa motum globi in tubo hab ich auch solvirt...” then, we will examine his solution.

In Fig.3, when a tube $AD$ including a small sphere rotates around a point $A$ on a horizontal plane, he tries to determine velocities of the tube and the sphere and a curve described by the sphere.

Moveatur tubus $AD$ continens globum $F$ super plano horizontali circaolum $A$, sitque determinanda curva, quam describet globus una cum velocitatibus globi et tubi. 22

At the first instant, the tube and the sphere are at rest at $AD$ and $B$, respectively. Then, a force acts on the tube perpendicularly and after that, no external force acts on it. A curve $Bnm$ represents an arc described by the point $B$ around the point $A$. Now, the tube and the sphere proceed to $AE$ and a point $a$, respectively. After an infinitesimal instant $dt$, the tube and the sphere go to $AF$ and a point $p$, respectively, and $op$ represents an infinitesimal line described by the sphere during $dt$. He supposes that after arriving at the point $p$, the sphere is free from constraint of the tube, then it will describe an infinitesimal prolonged line $pd (=op)$ during a succeeding time $dt$. And the tube will proceed to $Ab$ without interaction. Here, an arc $nm$ is equal to an arc $mg$. By drawing an arc $ac$ and setting $AB=a$, $Bn=x$ and $ Ao=y$, he obtains the relation $da = 2dx \sqrt{y/y \cdot a}$ 23. However, in reality, the tube and the sphere interact during the second instant $dt$. To explain this situation, he assumes a force (potentia) and the sphere is pressed by it against the tube and another force acts on the tube at the point $a$ and presses it against the tube. As a result, they will meet at a point $c$. 24 Namely, in reality, the tube and the sphere will be $Acf$ and the point $c$, respectively after the second infinitesimal instant $dt$. Assuming that $m$ stands for a mass of the sphere, $M$ for a mass of the tube, $d$ for a distance from the point $A$ to the center of gravity of the tube, $D$ for a distance from the point $A$ to center of oscillation, he obtains “erit ex mechanici $ac : dc = m : \frac{dD}{yy}$ 25.” From these equations, he obtains
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\[
ac = \frac{m\gamma y}{m\gamma y + Md^2} \cdot 2\frac{dx}{dy} \quad \text{and} \quad dc = \frac{Md^2}{m\gamma y + Md^2} \cdot 2\frac{dx}{dy} \cdot \frac{ydx}{ads} = \frac{ddx}{ds}.
\]

Next, by drawing a line perpendicular to a line \(pc\), he obtains \(ec = \frac{Md^2}{m\gamma y + Md^2} \cdot 2\frac{dx}{dy} \cdot \frac{ydx}{ads} = \frac{ddx}{ds}\). Here, he puts \(op\) as \(ds\). He also obtains \(hg = \frac{m\gamma y}{m\gamma y + Md^2} \cdot 2\frac{dx}{dy} = -\frac{dx}{V}\). By assuming that a velocity of the point \(B\) is \(c\) and that of the point \(n\) is \(V\) and from the equation \(ddx = dV \cdot dx/V\), because of \(dt = dx/V\), and the relation for \(hg\), he obtain the relation \(\frac{2m\gamma y}{m\gamma y + Md^2} = \frac{dV}{V}\). And integration of this relation gives \(V = \frac{maa + Md^2}{m\gamma y + Md^2} \cdot c\). Therefore, he obtains \(dx = Vdt = \frac{maa + Md^2}{m\gamma y + Md^2} \cdot cdt\) and substituting the relation for \(ec\), this relation gives \(\frac{2ydy}{(m\gamma y + Md^2)^2} = \frac{Md^2(maa + Md^2)^2}{ccdt^2}\).

By integrating this relation, he obtains \(\frac{2m(m\gamma y + Md^2)^2}{2Md^2(maa + Md^2)^2} = \frac{Mdd^2}{ccdt^2} - C\), where \(C\) is a constant. Representing a absolute velocity of the sphere at the point \(o\) by \(u\) and supposing that at the point \(B\), \(u\) is equal to \(c\) and the sphere does not have any velocity component along the direction \(AD\), because of \(dt = ds/u\), he obtains

\[
u = c \sqrt{\frac{maa + Md^2}{maa} - \frac{Md^2(maa + Md^2)^2}{maa (m\gamma y + Md^2)}}.
\]

A equation of a curve described by the sphere is given by the relation \(V : u = dx : ds\).

Finally, he gives a pressure of the sphere caused by the tube by \(\frac{4V}{a} \cdot \frac{dy}{dx} \cdot \frac{Md^2}{m\gamma y + Md^2} \cdot m\).

§.3-2 Analysis in his “Nouveau Probleme de Mecanique” (Daniel Bernoulli)

In this article, Daniel Bernoulli defines a circular motion of a body as le mouvement circulaire and its velocity along a tangent to the circle as la vitesse circulaire in §.I. In §.II, he defines a normal motion caused by circular one as le mouvement centrifuge and its velocity along a normal direction as la vitesse centrifuge. In §.III, he defines the product of la vitesse circulaire, a mass of the body and a distance from the body to the center of rotation as le momentum du mouvement circulaire. This quantity corresponds to angular momentum. In §.IV, he shows that even a body proceeding on a straight line \(BD\), has le mouvement circulaire and le mouvement centrifuge, if we consider it from a point \(A\) (Fig.4)
Next, he considers a tube as an extended body and he describes that the momentum du mouvement circulatoire of a tube rotating uniformly around a fixed point is conserved in §.V and gives its quantity in §.VI. Representing a distance from a fixed point to an given point of the tube by \( y \), la vitesse circulatoire of the given point by \( V \), a distance from the fixed point to any point of the tube by \( x \), an infinitesimal element of a distance \( x \) by \( dx \) and an infinitesimal mass element of the tube by \( d\xi \), he gives le momentum du mouvement circulatoire by \( V/y \times \int xx d\xi \) or \( MdDV/y \).

Moreover, in §.VII, he introduces an important lemma.

Si l’on applique au tuyau mobile autour d’un point fixe une puissance dans un point quelconque dont la distance au point fixe soit \( y \), cette puissance produira la même acceleration ou le même retardement sur le tuyau, qu’elle produirait si le tuyau n’avait point de masse & qu’il y eut dans le point, où la puissance est appliquée, une masse concentrée, qui fut

\[
\frac{dD}{yy} = M^{28}
\]

He wrote in §.IX, Proposition fondamentale as below.

Si un tuyau droit renfermant un globe librement mobile tourne sur un plan horizontal autour d’un point fixe, je dis qu’il y aura toujours le même \textit{momentum} du mouvement circulatoire dans le Systeme du tuyau & du globe.

A sphere is included at a point \( B \), when a tube has a position \( AD \) (Fig.5). At any instant, the tube and

![Fig.5](image)

the sphere are at \( Ai \) and at a point \( o \), respectively. To determine motions of the tube and the sphere, he considers them as follows. He assumes that during an infinitesimal time \( dt \), the tube proceeds from \( Ai \) to \( AF \) and the sphere proceeds from the point \( o \) to a point \( p \) and during the succeeding infinitesimal time \( dt \), if the sphere becomes free from constraint of the tube, the tube and the sphere will proceed from \( AF \) to \( Ab \) and the point \( p \) to a point \( d \), respectively. Consequently, \( \pi AF \) is equal to \( \pi Ab \), and \( op \) and \( pd \) are equal and on a straight line. From §.IV and V, he concludes that le momentum du mouvement circulatoire of the tube and the sphere are conserved respectively during free motion.

However, in reality, the tube and the sphere interact during the second time \( dt \).

...or cette action consiste à refîner le globe d avec le tuyau Ab, & comme cette action ne sçairoit se faire que perpendiculairement au tuyau, on tirera la petite da perpendiculaire à Ab, & puis on conçevra cette da comme un fil attaché par ses deux bouts au globe & au tuyau, qui se resserre entièrement jusqu’à ce que les deux bouts viennent à se toucher au point c, par là on voit qu’au bout du second element de tems la vraie situation du globe sera en c & celle du tuyau en Acf ;...

The editor of the complete works of Daniel Bernoulli named this method the ‘Daniel Bernoulli-
d’Alembert’s principle’. By putting \( Ac = y \) and supposing that the others have the same meaning as §.3-1, constriction of a string \( da \) has the same effect when the tube is weightless and instead has a concentrated mass \( dD/yy \times M \) at the point \( a \) based on the discussion in §.VIII. Therefore, he obtains the same relation as in §.3-1, \( ^{29} \)

\[
ac \cdot ac = dD/yy \times M : m.
\]

Consequently, he describes that the increase of le moment du mouvement circulatoire of the sphere is equal to the decrease of le moment du mouvement circulatoire of the tube and that le moment du mouvement circulatoire of the whole system has the same value at each instant.

Moreover, he proves that the increase of la vitesse circulatoire of a body during \( dt \) (\( \sim dv \)) are represented by

\[
dv = \frac{VV}{y} \times dt
\]

in both cases, where the body moves uniformly (§.XII) or under the influence of central force (§XIII). Here, \( V \) represents la vitesse circulatoire of the body at that instant and \( y \) represents a distance from a fixed point in §.XII and a distance from a center of force in §.XIII. Consequently, in §.XIV, he demonstrates that when a straight tube including bodies in it rotates around a fixed point, the increase of la vitesse circulatoire of each body is \( \frac{VVdt}{y} \).

Then, he transfers to Problème principal in §.XIV.

Soit sur un plan horizontal un tuyau droit AF renfermant tant de Corps qu'on voudra en B, C &c. & soit d'abord tout le systeme en repos ; qu'on s'imagine ensuite que le tuyau reçoive une impulsion & commence à tourner avec une vitesse donnée autour du point A ; il est question de déterminer à chaque moment & à chaque situation du tuyau le mouvement de toutes les parties du système.

In Fig.6, small bodies included in a tube proceed from a point \( E \) to a point \( I \) and a point \( D \) to

a point \( g \), respectively during an infinitesimal time \( dt \). \( Eh, Df \) and \( Bmn \) represent arcs and \( m, mN \) represent masses of each body \( B, C \), respectively. Moreover, \( p, pN \) represent les vitesses centrifuges of each body, \( a, aN \) represent distances of each body from a point \( A \) to each body on \( AF \) and \( x, xN \) represent distances of each body from the point \( A \) to each body on \( Ag \). In addition, \( y \) stands for an arc \( Bm, dy \) stands for an arc \( mn \) and \( C \) stands for la vitesse circulatoire at a point \( B, V \) stands for la vitesse circulatoire at a point \( m \).

\[\text{Le moment du mouvement circulatoire of a whole system in the first instant is equal to} \]

\[
\frac{MdD + maa + m'd'a' + \ldots}{a} \cdot C
\]
and when the tube proceeds to ADEG, le momentum du mouvement circulatoire of the whole system is equal to

$$\frac{MdD + mxx + m'x'x' + \ldots}{a} \cdot V.$$  

Because le momentum du mouvement circulatoire of the whole system is conserved, he obtains

$$V = \frac{MdD + maa + m'a'a' + \ldots}{MdD + mxx + m'x'x' + \ldots} \cdot C.$$  

From §.XIV, he obtains

$$dp = \frac{xVV}{aa} \cdot dt$$  

and because of \( p = \frac{dx}{dt} \), he obtains

$$aapdp = VVx dx.$$  

Substituting the value of \( V \) for this equation, he gives

$$aapdp = \left( \frac{MdD + maa + m'a'a' + \ldots}{MdD + mxx + m'x'x' + \ldots} \right)^2 CCx dx.$$  

In addition, substituting

$$x' = \frac{a'}{a} x, \quad x'' = \frac{a''}{a} x,$$

for the equation above and integrating it, he obtains

$$pp = \frac{(MdD + maa + m'a'a' + \ldots)(xx - aa)}{MaadD + (maa + m'a'a' + \ldots)xx} CC.$$  

Les vitesses centrifuges of each body, \( p, pN.. \) are determined by

$$p' = \frac{a'}{a} p, \quad p'' = \frac{a''}{a} p,$$  

Then, he obtains

$$V = \frac{MdD + maa + m'a'a' + \ldots}{MaadD + (maa + m'a'a' + \ldots)xx} aaC.$$  

and gives a time, which the tube proceeds from ABCF to ADEG, by

$$\int \frac{dx}{p}.$$  

Finally, by using two relations \( ma + mNaN + \ldots = s\delta \) and \( ma^2 + mNaN^2 + \ldots = e\delta s \), \( s = m + mN \ldots; \) \( \delta = \) a distance from the center of gravity to the point \( A; e = \) a distance from the center of oscillation to the point \( A \), he transforms the three equations derived above as bellow.

$$pp = \frac{(MdD + s\delta\delta)(xx - aa)}{MdDaa + s\delta\Delta xx} CC$$  

$$dy = aad dx \times \frac{\sqrt{MdD + s\delta\delta}}{\sqrt{(MdDaa + s\delta\Delta xx)(xx - aa)}}$$  

dt = \frac{dx\sqrt{MdDaa + s\delta\Delta xx}}{c\sqrt{(MdD + s\delta\delta)(xx - aa)}}$$  

Daniel Bernoulli solves these three equations in two extreme cases. The former is that the mass of the whole bodies can be neglected by comparing with that of the tube39 and the latter is that conversely mass of the tube is negligible by comparing with that of the whole bodies. Finally, he finishes his memoir by describing the conservation of vis viva in general.
§.4 Motions of a rotating tube including a body – Clairaut’s analysis.

Clairaut discusses motion of a tube including a body in his memoir. In §.XXIII, he poses the following problem (Fig.7) and solves it.

![Diagram of a rotating tube](image)

Soient sur un plan horizontal deux poids P & M, attachez à une ligne inflexible PCM, mobile autour du centre C; le premier des deux poids étant fixé sur la ligne inflexible, & l’autre pouvant glisser, on demande la courbe que décrit le corps M lorsqu’on donne une impulsion quelconque au corps P.

He assumes a system constructed by a weightless rod and two material points places at the both ends of the rod on a horizontal plane. One of the material points is fixed at one of the ends, the other is movable on the rod and the rod can rotate around a fixed point on the rod. First in §.XXIII-XXVIII, he discusses motions of the rod and the movable material point after an impact acted on the fixed material point. Next in §.XXIX-XXXII, he discusses a case, where the rod rotates on a vertical plane. In §.XXIII-XXXVII, and in §.XXVIII, he discusses cases, where the rod with two movable material points rotates on a horizontal plane and on a vertical plane, respectively. In §.XXXIX, he discusses a case, where the rod with any number of material points rotates on a horizontal plane and finally, in §.XL, he discusses a case, where the rod with any number of material points moves freely, not around a fixed point. Here, it is sufficient to examine the case discussed in §.XXIII-XXVIII. Clairaut gives two methods to solve this case. Let us examine his methods briefly.

§.4-1. Solution using the principles of la forces accélératrices and the conservation of vis viva (Clairaut a).

He puts $CP=p$, $PP=dx$, $CM=y$ and $Rm=dy$ and represents an infinitesimal duration, which the rod proceeds from $P$ to $p$ or from $M$ to $m$, by $dt$, mass of movable material point $M$ by $1$ and mass of fixed material point $P$ by $m$, respectively. Then, a velocity of the body $M$ along the direction $CM$ is equal to $y \cdot dx^2/ dt^2$, “par le principe général des forces accélératrices” he obtains

$$\left(\frac{ydx^2}{dt^2}\right) \frac{dy}{dt} = \frac{dy}{dt} \cdot \frac{ddy}{dt}.$$  \(34\)

Therefore, he obtains

$$ddy = ydx^2 ... (4-1-1).$$

Next, from the principle of conservation of vis viva, he gives

$$yydx^2 + dy^2 + mdx^2 = adt^2 \quad (4-1-2).$$

By differentiating (4-1-2) and substituting (4-1-1), he obtains

$$(m + yy)ddx + 2ydydx = 0.$$
and by integrating, he obtains

\[(m + yy)dx = bdt.\]

By using (4-1-2), he finally obtains

\[dx = \frac{dy}{\sqrt{(m + yy)(yy + m)ab - 1}}.\]

§.4-2. Solution using another principle (Clairaut), Clairaut states the principle used here as below.

**ARTICLE IV.**

**PRICIPÉ GÉNÉRAL ET DIRECT**

*Pour résoudre le Problèmes où il s’agit de déterminer le Mouvement de plusieurs Corps qui agissent les uns sur les autres, soit par des fils, soit par des leviers, soit de toute autre manière qu’on voudra.*

§. XXII.

Je commence par imaginer le système dans une situation quelconque, & je trace chacune des petites droites que les corps parcourent dans un instant ; je place ensuite au bout de ces petites droites, celles que les mêmes corps décriront l’instant d’après s’ils étoient libres : cela fait, je marque sur les directions suivant lesquelles les fils, les leviers ou autres instrumens agissent, de petites droites qui doivent exprimer les forces de ces instrumens, & que je détermine par cette condition que les diagonales des parallélogrammes faits sur ces petites droites & sur les prolongemens des côtés parcours par les corps dans le premier instant, soient terminées par des points où les corps étant supposé dans le second instant, les fils ou les leviers n’auroient souffert ni extension ni inflexion. Ayant par cette méthode deux côtés consécutifs quelconques de chacune des courbes décrites par les différents corps qui composent le système donné, la manière de trouver les équations de ces courbes n’est plus qu’une affaire de calcul. Ce principe, ainsi que les précédens, sera éclairci dans l’article suivant.36

Méli named this principle ‘d’Alembert’s principle’.37 Before solving this problem, he proposes two lemmas as preparation.

**LEMME I.**

*Mm & mn étant deux droites infiniment petites & égales, prise sur la droite HO ; MC, mC, nC trois droites tirées au point fixe C, je dis que si on nomme CM, y ; Cm, y+dy ; & l’angle MCM, dx ; on aura l’angle mCn=dx-2dydx/y.*38

We can demonstrate this lemma as follows (Fig.8). Because triangle MRm is congruent to

![Fig.8](image)

triangle mTn, we get relations MR=y⋅sin dx,y⋅dx=mT, mR=y+dy-v(y²-y²dx²).dy+y/2⋅dx².dy. And
because triangle $iCM$ is similar to triangle $iNT$, we obtain the ratio $(y+dy) dy = (ydx+it) iT$. By ignoring the infinitesimal smaller than the third order, we get the relation $iT(y+2dy) = ydx dy$. Finally, we obtain $iT = ydx dy(y+2dy) = ydx dy - 1/(y+1+2dy) dx dy(1-2dy/y). dx dy$. Therefore, we obtain $m = mT iT = ydx dy$. Then, we reach the final relation $mCn = (ydx dy) / (y+1+2dy) = (ydx dy/y) - 1/(y+1+2dy) + dx dy y dx$. because of $\tan \pi mCn \pi mCn = (ydx dy(y+dy))$. Here, we ignore the infinitesimal smaller than the third order.

**LEMME II.**

*Les même chose étant posées, je dis que* $Cn = y+2dy + ydx^2$.

We can demonstrate this lemma as follows. As stated above, triangle $mTmn$ is congruent to triangle $mTmn$. Because we can regard that $nT$ is equal to $ni$, we get the relation $iS = mS = nT$. Furthermore, we obtain $iS = mS/C$ because triangle $CSm$ is similar to triangle $mSi$. Next, we get the relation $iS = y^2 dx^2 / (y+dy)$. $ydx^2 dy + dx dy dx^2 dy = ydx^2 dy dx^2 dy = ydx dy dx^2 dy$, because of $mS = \pi mCS \times C = (dx dy y dy) + ydx dx dy y dx$. Here, we ignore the infinitesimal smaller than the third order. Finally, we obtain the relation $Cn = CS + iS + in = y+dy + ydx^2 + dy = y+2dy + ydx^2$.

Next, in § XXVIII, he solves the problem. In Fig 9, a rod rotates around a point $C$ and proceeds to $PCM$ from $ACB$. $P$ and $M$ stand for material points and $P$ is fixed on the rod but $M$ is movable on the rod. He puts $CP = l$ and $CM = y$ and supposes that $P$ proceeds to $p$ and $M$ proceeds to $m$ during the infinitesimal time $dt$ and puts $\pi PC = \pi MCm = dx$. During the succeeding time $dt$, he supposes that if two bodies do not interact, the body $P$ will describe $pr$ and the body $M$ will describe $mn$, here $r$ is on an arc $AP$ and $mn$ is on the prolongered line $Mn$. However, in reality, bodies are constrained by the inflexible rod and he represents this constraint to the body $P$ and to the body $M$ by impacts $pi$ and $mo$, respectively. These impacts are perpendicular with respect to the rod $mCP$ and have opposite directions mutually. In reality, the body $M$ proceeds to a point $\mu$, which is a vertex of a parallelogram $moum$ constructed by lines $mn$ and $mo$ and the body $P$ proceeds to a point $\pi$, where a prolongered line $C\mu$ and the arc $AP$ intersect, at the end of the second infinitesimal time $dt$. Consequently, $pi$ is equal to $\pi r$.

By supposing that a point $q$ is an intersection of the prolongered line $Cm$ and the arc $AP$, $qy$ is equal to $2dx dy/y$ from the lemmas and because $\pi r$ is equal to $-dx$, $qy$ is equal to $2dx dy/y$. Consequently, $qy = -mo$ is equal to $ydx + 2dx dy$. Therefore, since according to him, “...par les principes connus, la force $pi$ ou $\pi r$, multipliée par $CP$ & par la masse $P$, doit faire la même quantité que le produit de la force $mo$ par $Cm$ & par la masse $M$, he gives $\pi r \times CP \times P = mo \times Cm \times M..(4-2-1)$. And from (4-2-1), he obtains $yydx + 2ydx dy = -mdx$..(4-2-2)

and by integration, he gives $(yy+mo)dx = bdt ..(4-2-3)$. 

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where $bdt$ is a integral constant.

Next, because $\eta\mu$ is perpendicular to $Cn$, $Cu$ is equal to $y+2dy+ddy=Cn$ and from the lemmas, it is equal to $y+2dy+ydx^2$. Therefore, he obtains

$$ddy = ydx^2 \ldots (4-2-4)$$

By solving the equation (4-2-2) for $dx$ and substituting the result for (4-2-3), he gives

$$2dyddy = \frac{2bbydxdx^3}{(yy + m)^2}.$$  

By integration, he obtains

$$dy^2 = \frac{-bbd^2}{yy + m} + adt^2,$$

where $adt^2$ is a integral constant.

Eliminating $dt$ by using the equation (4-2-4), he obtains the final equation

$$dx = \frac{dy}{\sqrt{(m + yy)\sqrt{(yy + m)a/bb - 1}]].$$

which coincides with the final result obtained in §4.1.

§5 Motions of a rotating tube including a body – D’Alembert’s analysis.

D’Alembert poses the problem 2 (Fig.10) in Traité de Dynamique as below.

![Fig.10](image)

*Supposons qu’une verge GA fixe en G & située sur un plan horizontal, soit chargée de deux corps A, D, dont l’un A soit fixement attaché à la verge, l’autre D puisse couler librement le long de la verge par le moyen d’un anneau ; on demande la vitesse de chacun de ces corps à instant, & la courbe décrite par le corps D.*

D’Alembert also proposes the next lemmas to solve the problem (see Fig.11)

![Fig.11](image)
LEMME VII.

90. Si deux lignes infiniment petites $Pp$, $Mm$, (Fig.23) sont jointes par les lignes finies $PM$, $pm$, & qu'on fasse $px=Px$, & $my=My$ ; je dis 1°. que l'exces de $PM$ sur $pm$ est egal à deux fois la difference de $PM$ à $pm$, mois le quarré de l'angle fait entre $PM$ & $pm$, multiplié par $PM$.

2°. Que l'angle de $pm$ avec $pm$ est egal à l'angle de $PM$ avec $pm$, multiplié par $1+2.(PM\cdot pm)/PM$.

He further proposes a corollary as a special case of this lemma (Fig.12).

COLLAIRE II.

92. Si les lignes $Mm$, $my$ (Fig.24) sont $=0$, alors $Mx-PM=2pO+PO^2/PM$, & l'angle $pM-xPp=2pO/PM\cdot PMp$.

Using this corollary, d'Alembert solves the problem. Since Craig C. Fraser explained d'Alembert's method in detail, we will show it briefly.

D'Alembert supposes that during the first infinitesimal instant $dt$, a body $A$ describes an arc $AB$ and a body $D$ describes a line $DE$, and that during the second infinitesimal instant $dt$, if a rod does not disturb the motions of the two bodies, the body $A$ describes an arc $BC = \text{the arc AB}$ and the body $D$ describes a line $Ei = \text{the line DE}$. However, since the position of the body $D$ changes with time, the infinitesimal time $dt'$, which the body $A$ describes the arc $BC$ in reality, is different from $dt$. He designates the distance which the body $A$ describes uniformly with a velocity at the point $B$ during $dt'$ as $BQ$. Similarly, he signifies the distance, which the body $D$ describes uniformly with a velocity at the point $E$ as $Eo$. However in reality, since the bodies are restricted each other, at the end of the $dt'$, he supposes that the body $A$ proceeds to the point $C$ and the body $D$ proceeds to the point $P$.

Considering the motion $BQ$ of the body $A$ during the second instant as composed of the motion $BC$ and $CQ$, and the motion $Eo$ of the body $D$ during the second instant as composed of the motion $Ep$ and $EI$ (=op and perpendicular to the rod $GB$), we can realize that the rod will equilibrate when the bodies $A$ and $D$ have the unique motions $CQ$ and $EI$, respectively (d'Alembert's principle). Therefore,

$$A-CQ-GA=DE-GE,...(5-1)$$

By putting $GA=a$, $AB=dx$, $GD=γ$, $FD=dy/a$, $FE=dy$ and $CQ=a$, and using the relations $Mγ-PM=2pO+PO^2/PM$ and $(5-1)$, we obtain

$$α=2Dydy/dx/(Aa^2+Dy^2)....(5-2).$$

Next, by using the relation $γPM-γPp=(2pO/PM)γPp$, we obtain an equation of a curve described by the body $D$

$$ddy=dy+ydx^2/a^2+ady/dx....(5-3).$$

Finally, d'Alembert puts $dx=ady/a$ to simplify this differential equation and substitutes it for $(5-3)$. Furthermore, by integrating it, he obtains the next equation as a curve described by the body $D$. 
\[ dx = \frac{pdy}{a} = adyvD/v(Aa^2 + Dy^2)[2GD(Aa^2 + Dy^2) - 1] \ldots (5-4) \]

Here, \( G \) is an integral constant. The equations given by Clairaut (4-2-2) and (4-2-4) correspond to (5-2) and (5-3), respectively.

In the second edition, in Remarque II for the problem II, he discusses motion of a rod and a movable body, where the rod rotates on a vertical plane under the influence of gravity. And he discusses their motion, where the rod rotates with two movable bodies in Remarque IV and with three movable bodies in Remarque V on a horizontal plane. Moreover, he discusses their motion, where it rotates with one fixed body and two movable bodies in Remarque VI and with two fixed bodies and two movable bodies in Remarque VII. We discuss his solution and its relation with Clairaut’s solution in detail in the preceding article. In a word, although their metaphysical beliefs are quite different, his method is quite the same as Clairaut’s method from a physical point of view.\(^{46}\)

§.6 Motions of a rotating tube including a body — Euler’s analysis.

Euler discusses motion of a tube including a small body in his memoir E86. Here, we will describe his method briefly for comparison with Johann Bernoulli’s and Daniel Bernoulli’s methods.

In §.46, a straight tube \( OC \) is movable around a fixed point \( O \) (see, Fig.13). After a lapse of time \( t \), the tube rotates to a position \( OE \) and a body in the tube is on a point \( P \). Euler represents a mass of the tube by \( M \), moment of inertia of the tube with respect to the point \( O \) by \( Mkk \), a mass of the body by \( A \), \( \pi COE \) by \( \omega \), a length of the tube \( (OC=OE) \) by \( f \), a distance \( OP \) by \( x \) and an arc \( CE \) by \( s \).

Here, as a moment of force of the tube, Euler gives

\[ \frac{2Mkkdd\omega}{dt^2} \]

Next, he draws a perpendicular \( PQ \) to a line \( OC \) and puts \( QQ=p, QP=q \) and he represents a velocity component of the body along the direction \( Pr \) by \( dp/dt \), along the direction \( Pq \) by \( dq/dt \). Consequently, forces acting on the body along two directions can be expressed by \( 2Adp/dt^2 \), \( 2Addq/dt^2 \), respectively. By decomposing these forces along the directions \( PE \) and \( PL \), and expressing \( p \) and \( q \) by \( x \) and \( \omega \), he gives force along the direction \( PE \) by

\[ \frac{2Addx - 2Axd\omega}{dt^2} \]

and along the direction \( PL \) by

\[ \frac{4Adxd\omega + 2Axdd\omega}{dt^2} \]
In §.48, Euler solves a specific problem. He considers motions of a tube and a small body include in it after any force having acted on a tube only at the first instant. He represents a pressure of the body acting on the tube perpendicularly by $P$ and obtains the next three equations.

$$
\frac{2Ad\delta x - 2Axd\omega}{dt^2} = 0 \tag{6-1}
$$

$$
\frac{4Ad\delta x + 2Axd\omega}{dt^2} = -P \tag{6-2}
$$

$$
\frac{2Mkkdd\omega}{dt^2} = Px \tag{6-3}.
$$

By substituting (6-3) for (6-2), he gives a pressure $P$ by

$$
P = -\frac{4AMkkd\delta x d\omega}{(Mkk + Axx)dt^2}.
$$

By eliminating $P$ from (6-2) and (6-3), and integrating it, he obtains $d\omega = \frac{Eadt\sqrt{a}}{Axx + Mkk}$, where $Eadt\sqrt{a}$ is an integral constant. Substituting this equation for (6-1), multiplying $2dx$ and integrating it, he obtains $dx^2 = -\frac{E^2 a^3 dt^2}{A(Axx + Mkk)} + bdt^2$, where $bdr^2$ is an integral constant. From this relation, finally he gives

$$
dt = \frac{dx\sqrt{A(Axx + Mkk)}}{\sqrt{bA(Mkk + Axx) - E^2 a^3}}
$$

and

$$
d\omega = \frac{Eaadx\sqrt{A}}{\sqrt{(Mkk + Axx)\left[Ab(Mkk + Axx) - E^2 a^3\right]}}.
$$

Next, Euler proves that in this case, vis viva of the whole system is conserved in §.49 and he considers the motion of the body in the tube and that of the tube when the tube rotates around a fixed point on a vertical plane under the influence of gravity in §.50. In §.51, he considers a case, where $M$ is equal to zero. In §.52-60, he extends §.48 and considers a case where the tube includes three small bodies or further and he adds the condition that $M$ is equal to zero in §.61-63. Finally, he discusses the motion of a curved tube including a small body in it under the influence of no external force in §.68-72.

§.7 Discussions.

We have arranged the characteristics of the resolution methods used by the five scholars in the table. First, we realize similarities between Johann Bernoulli’s, Clairaut's, d’Alembert’s and Daniel Bernoulli’s methods, namely similar lemmas or relations. Johann Bernoulli proposes the equation (2-1) (see, Fig.1), Daniel Bernoulli proposes the equation $da = 2dxdy/\alpha$ (see, Fig.3), Clairaut proposes the equation $\pi mCn = dx - 2dydx/y$ (see, Fig.8). D’Alembert also proposes the relation $\pi(pMx-PMp) = 2pO/PM.PMP$. We can prove that their equations are quite the same as follows.

First, in Johann Bernoulli’s equation (2-1), by putting $p^2 + s^2 = y^2$ to use Clairaut’s notation and integrating it, we obtain $y\delta y = sds$. From §.2, we obtain $d\theta = \frac{pds}{p^2 + s^2}$ and according to Clairaut, we can put $d\theta = dx$. Then we can transform Johann Bernoulli’s equation
\[ ACB - BCD = \frac{2pds^2}{(pp + ss)^2} \] to Clairaut’s equation \( \frac{nm}{Cn-dx} = -2dy/dv/y \). Second, in Fig.3, we can express \( \pi dA = da/Ad = 2dx dy/\pi \times 1/Ad \). However, according to Clairaut’s notation, we can express \( Ad \) as \( y^2 + 2dy + dv \). Since the quantity \( dx/a \) in Daniel Bernoulli’s equation corresponds to \( dx \) in Clairaut’s equation, by representing \( 2dx dy/\pi \times 1/Ad \) by Clairaut’s notation and ignoring more than the 3rd order infinitesimal, we can prove that \( \pi dA = 2dy/\pi = dx - nmCN \), namely Daniel Bernoulli’s equation coincides with Clairaut’s. Needless to say, Clairaut’s and d’Alembert’s equations are very much the same. This is the case for their second equations, namely \( Cn = y^2 + 2dy + ydx^2 \) (Clairaut) and \( Mx - PM = 2po + P0^2/PM \) (d’Alembert).

These coincidences are based on the fact that their methods are essentially the same, especially we can realize that d’Alembert’s and Clairaut’s methods are quite the same, by comparing their equations in their lemmas. In the second infinitesimal instant, they regard the motions of the tube and of the body as if they move freely without interaction at the first step, and next, by considering an interaction of the tube and the body and adding this interaction to the free motions, they search for real motions.

Here, so-called d’Alembert’s principle is used to divide the motion caused by interaction. Daniel Bernoulli describes generally this principle in a constraint system as below in his article “Demonstrationes theorematum suorum de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae.”

Think that at a given instant the several bodies of the system are freed from one another, and pay no attention to the motion already acquired, since here we speak only of the acceleration or the elementary change of motion. Thus when any body changes its position, the system takes on a configuration different from that it would assume if not freed. Therefore imagine some mechanical cause to restore the system to its proper configuration, and, again I seek the change of position arising from this restitution in any body. From both changes you will learn the change of position in the system when not freed, and thence you will obtain the true acceleration or retardation of each body belonging to the system.

In this case, a real motion of the tube (or the body) can be expressed as the sum of a free motion of the tube (or the body) and the constraint motion of the tube (or the body) caused by the body (or the tube). This method corresponds to the principle Clairaut stated at the beginning of §4.2 (ARTICLE IV. PRICIPÉ GÉNÉRAL ET DIRECT). D’Alembert also proposes the similar principle. However there is still room to discuss whether their principles are quite the same.

Next, we will speak briefly about Johann Bernoulli’s method. Daniel Bernoulli, Clairaut and d’Alembert consider a case, where the tube sets to move by an external force in the first instant and later, the motion of the system is changed only by an interaction between the tube and the body including in it. In addition, Clairaut and d’Alembert consider the motion of the system under the influence of gravity. Euler considers this case, too. On the contrary, Johann Bernoulli solves only a case where the tube rotates uniformly. Consequently, this situation requires that the mass of the body is zero or an external force acts continuously to rotate uniformly and this requirement is quite different from the three researchers’. Moreover, Johann Bernoulli does not require the masses of the tube and the body in his solution and he solves the problem by using only geometrical relation. Then, when he writes that without a restriction of the tube, the body will proceeds to the point \( D \) during the second infinitesimal instant, but in reality since the tube proceeds to the point \( E \), the body proceeds to the point \( E \), too, at a first glance we tend to conclude that he uses so-called d’Alembert principle. However, this conclusion is wrong. As stated below, in d’Alembert’s principle, a motion of a body, which signifies the product of a mass of the body \( (m) \) and the infinitesimal change of a velocity of the body \( (dv) \) or the product of \( m \), \( dv \) and a distance from an axis of rotation to the body \( (r) \), must be considered but Johann Bernoulli never use such a quantity. Consequently, his method is old-fashioned even at that time.

Sometimes Truesdell claims that Daniel Bernoulli’s article (Daniel Bernoulli b) is the first to declare the conservation of angular momentum. Therefore, we will examine Clairaut b)’s and d’Alembert’s methods, which are equivalent to Daniel Bernoulli a)’s method. Furthermore, we will examine the relation between Daniel Bernoulli’s two methods. According to Truesdell, in 1703,
Jacob Bernoulli is the first to treat the conservation of angular momentum implicitly. In this article, he discusses the center of oscillation of a pendulum. Here, he supposes a quantity, which corresponds to the angular momentum of each mass element, and searches for the length of an equivalent simple pendulum of a composed pendulum by considering the equilibrium of these quantities. Then he invokes the principle of lever in statics on condition that ‘the motion itself be regarded as giving rise to forces per unit mass equal to the accelerations reversed.’ The quantity considered by him is ‘une branch d’un levier ‘H’une vitesse’ H’un poids ou une puissance’ and he regards the products as an extension of the principle of lever in statics, namely the products of the length of a lever and the weight. He declare that

Conjecture 1. The principle of moment of momentum, as an independent law of mechanics and as a generalization to kinetics of the principle of equilibrium of moments in statics, is due to James Bernoulli (1686, corrected 1703); in concept, though not in correct statement, it antedates Newton’s laws (1687). He continues by saying that this principle of conservation is explicitly shown in Daniel Bernoulli and Euler’s article.

... in 1745 both Daniel Bernoulli and Euler put forward a form of the principle in the course of their solutions of the problem of a mass point constrained to slide within a rigid, rotating tube. First, we will examine the methods of Clairaut, d’Alembert and Daniel Bernoulli, which precede that of Daniel Bernoulli and Euler.

Clairaut uses the relation (4-2-1) and it is certain that ‘principe connu’ signifies the principle of lever. However, a quantity considered by him is the products of the force, the length of a lever and the mass. Since he calls a force ‘la impulsion’ elsewhere, his quantity is proportional to angular momentum. D’Alembert states that his relation (5-1) signifies equilibrium of the products of the length of a lever, the mass and the motion, here motion means a velocity for him. Therefore, his products also correspond to angular momentum. However, their treatments are only an extension of the principle of equilibrium of lever. Furthermore, Daniel Bernoulli uses the relation ac:dc=m:DD:yy:M in Daniel Bernoulli. Contrary to d’Alembert and Clairaut, since he supposes that the tube has mass, DD:yy:M (une masse cocentrée) corresponds to a mass, which the mass of the tube are concentrated at a point o. And since distances ac and dc mean displacement during an infinitesimal time, namely velocities, ac:dc=m:DD:yy:M means equality of linear momentum of the material point and that of the tube. Therefore, by using the Newton’s third law, we can divide the line ac in this way, as shown by the editor of the complete works of Daniel Bernoulli. Since the length of the material point is equal to that of une masse cocentree, the relation of linear momentum emerges, but not that of angular momentum explicitly. Because in d’Alembert’s and Clairaut’s resolutions, quantities, which correspond to change in angular momentum during an infinitesimal time in case there are no restriction, vanish and in Daniel Bernoulli’s resolution, quantities, which correspond to that of linear momentum, vanish, the integral of this quantity over time is conserved. However, they do not state the products as a concept.

Next, we will examine the relation between Daniel Bernoulli’s two articles. Daniel Bernoulli solves the same problem by two methods as shown in §.3-1 and §.3-2. Then for the purpose of proving that these two methods are quite equivalent, we will demonstrate that the two methods give the same curve described by the tube. In §.3-1, he gives a relation V:u=dx:ds for the curve described by the tube. Because this relation is equivalent to another relation V:u=dx²:ds², we will substitute the values of V and u given by Daniel Bernoulli for the latter relation. Because of

\[ u^2 = c^2 \left( \frac{ma^2 + MdD}{ma^2} - \frac{MdD}{ma^2} \left( \frac{ma^2 + MdD}{my^2 + MdD} \right)^2 \right) \] \[ \text{and} \quad V^2 = c^2 \left( \frac{ma^2 + MdD}{my^2 + MdD} \right)^2 , \]

we obtain

\[ V^2 : u^2 = ma^2 \left( ma^2 + MdD \right) \left( my^2 + MdD \right) - MdD \left( ma^2 + MdD \right) \]

From \[ V^2 : u^2 = dx^2 : ds^2 \], we obtain

\[ dx^2 : ds^2 = ma^2 \left( ma^2 + MdD \right) \left( my^2 + MdD \right) - MdD \left( ma^2 + MdD \right) \]
Substituting \( ds^2 = \left( \frac{y}{a} \right)^2 + dy^2 \) for this proportion and solving it for \( dx \), we obtain

\[
dx = \frac{a\sqrt{Md + ma^2}}{\sqrt{my^4 + (Md - ma^2) y^2 - a^2 \cdot Md}} \cdot dy.
\]

By comparing this equation with \( dy = a^2 dx \times \frac{\sqrt{Md + ma^2}}{\sqrt{(Md Da^2 + ma^2 a^2) (x^2 - a^2)}} \), we can conclude that both equations are perfectly the same. And in the equation

\[
dy = a^2 dx \times \frac{\sqrt{Md + s \delta x}}{\sqrt{(Md Da^2 + s \delta x a^2) (x^2 - a^2)}} \],
\]

which Daniel Bernoulli gives in \( \S \).3-2, when the system has only one body, we obtain the latest equation. Finally, we can conclude that the both methods are perfectly the same.

Then, what is the difference between two methods? In his letter to Euler, Daniel Bernoulli writes as below.

Ich habe auch über dergleichen problemata einige compendia, sonderlich ratione vis acceleraticis globi in tubo, vermitelst welcher ich kann die differentialia 2 \( ^{46} \) gradus evitiren und die ganze Solution kürzer machen, die Zeit erlaubt mir aber nicht, solche nunmehro zu explieiren.\(^{58} \)

And in another letter to Euler, he writes as below.

Ew. Principium conversionis momentorum motus rotatorii abreviiit freyliglich die problemata de motu corporis in tubo: ich hatte aber solches auch schon observirt, und ist ein corollarium von der methodo directa, die ich Ihnen einmal für einen gewissen casum überschrieben hatte.\(^{59} \)

Therefore, according to him, his latter solution has its novelty in respect that he does not use the second order differential but conservation of le momentum du mouvement circulatoire.

First, we consider conservation of le momentum du mouvement circulaire. He uses the quantity \( dD/yyHM \), which we took up in \( \S \).3-2, in both methods. As described above, he derives this quantity in his preceding article “De Variacione motuum a Percussione excentrica.” According to the article, a plane \( ABC \) on a horizontal plane is rotatable around a point \( D \) (Fig.14). A force (potentia)

\[\text{Fig.14}\]

acts on a point \( B \), whose infinitesimal mass is \( \mu \), and the plane \( ABC \) rotates around the point \( D \) with acceleration. He searches for a relation between \( \mu \) and \( m \), which is an infinitesimal mass of a point \( E \) on the plane \( ABC \), when the point \( E \) has the same acceleration (acceleratio) as the point \( B \). The result is \( \mu \cdot m = DE^2 : DB^2 \).\(^{50} \) Here, we interpret according to the editor of complete works of Daniel Bernoulli, that he searches for a relation in which the point \( B \) and the point \( E \) have the same angular velocity (\( \omega \)). Then, by putting the increase of the point \( B \) during an infinitesimal time \( dt \) \((d\nu_B) = DB \cdot d\omega \), and that of the point \( E \) \((d\nu_E) = DE \cdot d\omega \), a quantity of the point \( B \), \( \mu \cdot DB \cdot d\nu_B = \mu \cdot DB^2 \cdot d\omega \) is equal
to a quantity of the point $E$, $m \cdot DE \cdot dv = m \cdot DE^2 \cdot d\omega$. This relation means that the change of angular momentum of the point $B$ with respect to the point $D$ is equal to that of the point $E$ with respect to the point $D$. By expansion, regarding the change of angular momentum of the point $B$ with respect to the point $D$ as equal to the change of angular momentum of the whole plane $ABC$ with respect to the point $D$, we obtain $\mu \cdot DB^2 \cdot d\omega = I \cdot d\omega$. Here, $I$ signifies momentum of inertia of the plane $ABC$ with respect to the point $D$. Therefore, we obtain $\mu = I/DB^2$, which corresponds to $dD/yy \times M$ described above.

Then, he states le momentum du mouvement circulatoire, which corresponds to conservation of angular momentum, in his “Nouveau Probleme de Mécanique” for the first time but already its idea can be recognized in his “De Variatione motuum a Percussione excentrica.” Therefore, we can understand that his solution given by his letter includes the concept of conservation of angular momentum implicitly.

In “Nouveau Probleme de Mécanique”, Daniel Bernoulli proves that le momentum du mouvement circulatoire of the tube and the body is conserved respectively when they move freely without constraint. Next, he proves that when considering an interaction, le momentum du mouvement circulatoire of the whole system is invariable. For this demonstration, instead of the tube whose mass distributes wholly, he considers an interaction of the body in the tube and the tube whose mass, $dD/yy \times M$, concentrates in the point $a$ (Fig.5). And he divides the line $ad$ into $ac : dc = m : \frac{dD}{yy} \times M$. Consequently, multiplying the impulse by a distance from the center of rotation $A$ to a point which an interaction acts, the increase of le momentum du mouvement circulatoire of one is naturally equal to the decrease of le momentum du mouvement circulatoire. As for le momentum du mouvement circulatoire, he states that “c’est à l’imitation de ce qu’on appelle momentum d’une force qui agit sur un levier.” It is that his concept of le momentum du mouvement circulatoire is an extension of the principle of lever.

Finally we can conclude that the difference between Daniel Bernoulli’s two methods is that he does not use the second difference in his “Nouveau Probleme de Mécanique.” Namely, his method stated in his letter uses the second difference and geometrical relations. The same thing can be said of Johann Bernoulli’s, d’Alembert’s and Clairaut’s methods in §4-2. These methods need a special technique in a specific problem and it is difficult to set equations of motion generally in any situation. On the contrary, the method in “Nouveau Probleme de Mécanique” does not need such a special technique and we can obtain equations of motion easily, just like Clairaut’s method. However, there exists the essential difference between giving a name as a notion and using unconsciously, though Clairaut makes an excuse!

This principle of conservation is pointed out by Euler and he further generalizes this principle. Truesdell states that the method used by Euler is the same as James Bernoulli’s, but it is clearly explained, and Euler gives (5) explicitly in the special case of plane motion about a fixed axis ($L = Ia$); his alleged proof is no more than an assertion of the extended law of the lever.

Here, the equation (5) signifies $d\mathbf{H}/dt = \mathbf{L}$ and $\mathbf{H}$ is angular momentum (or ‘the moment of momentum’) and $\mathbf{L}$ is ‘the total torque exerted by the external forces.’

When a body rotating around a point $O$ changes its motion by an external force (Fig.15), Euler

![Fig.15](image-url)
searches its motion by equating the sum of moment of force gained by each element of the body to moment of force given by an external force to the body. He searches for a force that acts on a point A, whose direction Aa is perpendicular to line OA and then multiplies this force by distance OA and sums up the products over the body. He names the final quantity the sum of moment of force (omnia harum virium summa momentorum ad axem). Then, by equating this sum to moment of force caused by an external force, he derives the equation of a rotating body. It is certain that his starting point is the principle of lever, because he considers moment of force, which is equal to the products of the length of a lever and force. However, the establishment of the principle of conservation of angular momentum from the principle of lever in statics is not simple. In discussing the history for searching the length of an equivalent simple pendulum of a composed pendulum, Christiane Vilain makes clear the process of development from Jacob Bernoulli’s quantity, which is the products of the length of a lever, the velocity and the weight derived from the principle of lever in statics, via Johann Bernoulli’s and Hermann’s quantity, to Euler’s quantity, which is moment of force with no relation to a lever. Therefore, Truesdell’s evaluation to Euler is too simple.

Furthermore, Euler gives a general equation for a rotating body in the two dimensions 2Mkkd\(\frac{d\theta}{dt}\)=P\(\theta\) (which corresponds to \(dH/dt=L\)). And Euler, contrary to Daniel Bernoulli, uses a concept of moment of inertia (momentum inertiae corporis\(=Mkk\)). Furthermore, he demonstrates that in this special case, namely the motion of a rotating tube including a material point, angular momentum (momentum motus gyroriori) of the system is conserved.

From another point of view, we can divide their methods into two groups; one includes Euler and d’Alembert, the other includes Daniel Bernoulli, Clairaut. Euler declares at the beginning of his article treating a body in a rotating tube that:

Quod negotium cum suscepsissem, praecepue in hoc elaboravi, ut huiusmodi problematum solutiones ex primis mechanicae principiis investigarem, neque ad hae principii indi
demum derivatis, cuiusmodi est conservatio virium virarum, uterum; non quasi huius
principii veritatem in dubium vocarem, sed potius ut consensus meum solutionum cum
iis, quae ex principiio isto sint deductae, eius veritas etiam illis, qui adhuc de eo dubitant,
plenissime confirmaretur. Tum vero saepe occurrunt casus, quibus hoc principium alias
utilissimum, omni usu ad solutionem obtinendum caret, eum contra prima mechanicae
principia, si recte adhibeantur, semper ad solutionem perduere debeant.

Namely, he uses the primary principles, not conservation of vis viva. He also writes in another article that:

...l’une [méthode], tirée des premiers principes de la Mécanique, par le moyen desquels on
trouve, à chaque instant, le changement tant dans la vitesse que dans la direction, causé
par les forces sollicitantes...Mais la première méthode, quoiqu’elle soit beaucoup plus
difficile que l’autre [fondée sur des principes dérivatifs]... Semble pourtant être plus
naturelle, parce qu’elle montre à chaque instant, non seulement l’état du mouvement, mais
encore les véritables causes de tous les changements qui arrivent successivement.

Next we will states his primary principles used in §.6 of this memoir briefly. In Fig.11, he sets
\(OQ=p, \quad OP=q\) and signifies a force acting on the body \(P\) along the direction \(OQ\) as \(F_{OQ}\), a force
acting along the direction \(QP\) as \(F_{QP}\) and firstly he gives the equations of motion for the body \(P\) by

\[
F_{OQ}=2A \cdot ddP/dt^2, \quad F_{QP}=2A \cdot ddq/dt^2
\]

and by signifying moment of inertia with respect to the point \(O\) as \(Mkk\), a rotational angle around the
point \(O\) as \(\omega\) and a moment of force acting on the tube as \(N\), secondly he gives the equation of
rotation of the tube by

\[
N=2Mkk \cdot d\theta/dt^2.
\]

It is these equations that are used at present time to solve such a problem

D’Alembert writes as below.

L’élegance dans la solution d’un Problème, consistant surtout à n’y employer que des
principes directs & en très-petit nombre, on ne sera pas surpris que l’uniformité qui regne
dans toutes mes solutions, & que j’ai eue principalement en vue, les rende quelquefois un
peu plus longues, que si je les avois déduites de principes moins directs.\textsuperscript{68}

For Euler and d’Alembert, mechanics must be deduced from primary principles, though their system are quite different. Clairaut, on the other hand, uses various methods to solve the problem. And as for Daniel Bernoulli, the editor of the complete works of Daniel Bernoulli writes that While Daniel Bernoulli used this principle for solving various problems, d’Alembert stated it as the universal principle of dynamics able to replace even Newton’s notion of force and his laws of motion. …here it suffices to say that Bernoulli never made such a claim.\textsuperscript{69}

Finally, we can place their methods as follows. Johann Bernoulli’s method treats a special case of a rotating tube including a body in it geometrically. We can not find a mass and motion of the body in it. Next, Daniel Bernoulli \textsuperscript{87}'s method, Clairaut \textsuperscript{83}'s method and d’Alembert’s method use the second differential and geometrical relations, which are based on the 17\textsuperscript{th} century tradition. In addition, the concept of une masse concentré introduced by Daniel Bernoulli, is his device to treat an extended body on the basis of the tradition of Huygens’ origin. Daniel Bernoulli \textsuperscript{81}'s method, which is based on this tradition derived from the 17\textsuperscript{th} century, introduces conservation of angular momentum and is new one. And Clairaut \textsuperscript{83}'s method is composed by a skillful combination of conservation of vis viva and $\phi dt = dv$ in the 18\textsuperscript{th} century fashion. However, their methods are proposed to solve given problems skillfully under the specific conditions and are not the primary principles applicable to all cases. On the contrary, the method given by Euler in §6 treats not only the special case, namely motion of a tube including a material point but also by introducing a concept of moment of inertia, a general method to solve motion of rigid body.

\textbf{notes}

\begin{enumerate}
\item Euler, \textit{Opera Omnia}, ser.4A, vol. 2, Einleitung Euler-Johann I Bernoulli, p.59. Unfortunately, this letter was missing.
\item See, Johann Bernoulli’s letters to Euler dated on March 15, on August 27 and on March, 1742. (Euler, \textit{Opera Omnia}, ser.4A, vol.2, Nr.33, 34, 35.)
\item Euler, \textit{Opera Omnia}, ser.4A, vol.2, Nr.33, p.413.
\item Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIème siècle, ed. P.H. Fuss, 2 vols. (St. Petersburg, 1843), Vol. II, Lettre de Daniel Bernoulli à Léonard Euler, Lettres XXII, XXIII, XXV, XXVII, XXVII, XXIX, XXX, XXXI, XXXII and XL.
\item Fuss, ibid. Lettre XXVI.
\item Euler, \textit{Opera Omnia}, ser.4A, vol.5, n°18, 19 and 20.
\item Clairaut, “Sur quelques principes qui donnent la solution d’un grand nombre de problèmes de dynamique”, \textit{Mémoires de l’Académie des Sciences de Paris}, (1742), 1745, pp.1-52.
\item Euler, “De motu corporum in superficiebus mobilibus”, Opuscula varii argumenti 1: \textit{Opera Omnia} ser.2, vol. 6, pp.75-174, (E86).
\item D’Alembert, \textit{Traité de Dynamique}, (Paris, 1\textsuperscript{er} ed. 1743 ; 2\textsuperscript{nd} ed. 1758), Problème II.
\item Here, we use a figure in Latin version, because the French version does not have a correct figure.
\item Exactly speaking, $EC$ is equal to $x+2dx+dx$, but by approximating $\pi DCE$ within 2\textsuperscript{nd} order
\end{enumerate}
infinitesimal, we can reach at this equation.

19 "n" means a natural logarithms "in".

20 In the Latin version, he deduces the relation $2dydx^3 = ds^3 \cdot d(xdy/ds)$ from $\pi DCE = \pi BCA - \pi DCB$. Then, by substituting this relation for $dy = xdx/a$ and considering that $dx$ is constant, he obtains $2xdx^3 = ds^3 \left( \frac{x^2}{ds} \right)$, namely $2dx \cdot ds^2 - xds \cdot dds = 2dx^3$. Next he substitutes this result for two equations $ds^2 = dx^2 + dy^2 = dx^2 + \frac{x^2dz^2}{a^2}$ and $ds \cdot dds = dx \cdot ddx + \frac{xdx \cdot dz^2}{a^2}$. Then he deduces $xdx \cdot dz^2 - b^2dz^2 = a^2dx^2$ or $dz = \frac{adx}{\sqrt{x^2 - b^2}}$, where $b$ is constant. By integrating this relation, he finally obtains $z = \int \frac{adx}{\sqrt{x^2 - b^2}} = a \log \left( x + \sqrt{x^2 - b^2} \right)$.

21 Fuss, ibid., lettre XXVI.

22 Fuss, ibid., lettre XXVI, p.500.

23 For derivation of this relation, see §6.

24 "Num vero concipiendae est potentia, quae globum versus tubum premet, et altera aequalis, quae tubo in a applicata, eundem premet versus globum; hoc modo globus et tubus convenient in c; et itaque position globi post secundum tempusculum dt in c, et positio tubi in Acf,...“(Fuss, ibid., lettre XXVI, p.501.)

25 For derivation of this relation, see §6.

26 We can demonstrate this relation as below. In Fig.16, because we can regard that a line $og$ is parallel to a line $cd$, $pog$ is equal to $pdfc$. Moreover, because a line $de$ is perpendicular to a line $ec$ and we can regards that a line $pd$ is perpendicular to the line $de$, we obtain $pog = pdfc$. Therefore, triangle $ogp$ is similar to triangle $ced$. Then, we obtain $og : op = ce : cd$ namely $ydx/a : ds = ce : dc$. Finally, we have a conclusion, $ec = cd = (ydx/ad) = MdD/(My + MdD) = (2dx/a)/(ydx/ad)$.

27 We can derive this relation as follows. A pressure of the sphere caused by the tube is equal to a perpendicular component to the tube $Af$ of $m \times dds/dt^2$, namely a component along the direction $dc$ (see, Fig.3). Because of the relation $ydx/a : ds = ce : dc$, a required pressure is $m \times dds/dt^2 \times ds/(ydx/a)$. By considering $dt = dx/V$, we obtain

\[
2V \cdot \frac{dy}{a} \cdot \frac{MdD}{myy + MdD} \cdot m.
\]

Finally, by considering a factor 2 caused by Euler’s unit, we obtain

\[
4V \cdot \frac{dy}{a} \cdot \frac{MdD}{myy + MdD} \cdot m
\] as a pressure.
As for this relation, see, §6.
Daniel Bernoulli, Werke 3, General Introduction, pXXVI.
We must pay attention to the notations $x$ and $y$, because their meanings are interchanged from §XVI.

Daniel Bernoulli derives this relation as follows. He obtains $x'dp = xdp'$ from $dp = \frac{xV^2}{a^2}dt$
and $dp' = \frac{x'V^2}{a^2}dt$. Because $dx$ and $dxN$ are proportional to $p$ and $pN$, respectively, he obtains $pdx' = p'dx$. By adding two equations, he gives $x'dp + pdx' = xdp' + p'dx$. And by integrating this equation, he obtains $x'p = xp'$. He derives $dx'/x' = dx/x$ from $x'p = xp'$ and $pdx' = p'dx$. Finally by integrating the latest result, he derives $x'/a' = x/a$, here a integral constant is determined adequately.

It is the case that is solved by Johann Bernoulli.

Clairaut, ibid.

We can derive this equation as follows. By representing la force accélératrice by $\varphi$, a velocity of a body by $u$ and a distance described by the body by $v$, we obtain $\varphi dt = du$ and then, $ede=udu$ (See, for example, J.L.d'Alembert, Traité de Dynamique, (2nd ed.), p.24). A force acting on the body along the direction CM is a centrifugal force and its quantity is equal to a velocity component at a point $M$ along the direction MR to the 2nd power over CM, namely $y dx^2/dt^2$, which is equal to $\varphi$. By substituting $de=dy, u=dy/dt$ and $du=d(dy/dt)=ddy/dt$ for $ede=udu$, we obtain the required equation.

Here, a stands for constant. The first term, the second term and the third term of the left side are quantities derive from a velocity of the body $M$ along the direction $MR$, along the direction $Rm$ and that of the body $P$, respectively.

Clairaut, ibid., pp.21-22.
Meli, ibid., p.307.

Clairaut, ibid., pp.24.
Clairaut, ibid., p.25.
Clairaut, ibid., p.25.
D'Alembert, ibid., p.104.
D'Alembert, ibid., p.102.
D'Alembert, ibid., p.104.

Craig G. Fraser, "D'Alembert's principle: the original formation and application in Jean d'Alembert's Traité de Dynamique," Centaurus 28. Copenhagen, 1985, pp.31-61(Part One) & pp.145-159(Part Two). Fraser derives the equations (5-1), (5-2) and (5-3) in his article in detail.

Apparent Clairaut solves the problem under the assumption that $dt$ is constant and d'Alembert supposes that $dt$ is variable. However, their methods are quite the same except for this condition.

According to the modern notation, this quantity corresponds to $I \times \frac{d\omega}{dt}$, where $I$ signifies a moment of inertia of the tube with respect to the point $O$ and $\omega$ signifies an angular velocity of the tube.

Translated by C. Truesdell (C. Truesdell, The Rational Mechanics, in Leonhardi Euleri Opera Omnia, ser.2, vol.10, pp.159-160.). The original sentences are:
"Puta in systemate ad momentum temporis corpora singula a se inuicem solui, nulla facta attentione ad motum iam acquisitum, quia hic de acceleratone seu mutatione motus elementari tantum sermo est : ita quilibet corpore situm suum mutate, systema aliam accept famiguram, quam non-solutum habere debeat : Igitur finge causam mechanicam quamcumque systema in debitam figuram
restituentem atque, rursus inquit in mutationem situs ab hac restitutione oram in quolibet corpore; et ex vtraque mutatione intelliges mutationem situs in systemate non soluto, indeque accelerationem retardationemue veram cuiusuis corporis ad systena pertinentis obtinebis."

D’Alembert, T.D., Problème general, pp.73-75.


53 C. Truesdell, ibid., III. Reactions of Late Baroque Mechanics to Success, Conjecture, Error, and Failure in Newton’s *Principia*, p.156.

54 C. Truesdell, ibid., p.252.

55 C. Truesdell, ibid., p.252-254.

56 D’Alembert, T.D., p.73.

57 Daniel Bernoulli, *Werke 3*, introduction, p.93.

58 Fuss, ibid., lettre XVII, p.511.

59 Fuss, ibid., lettre XXIX, pp.525-526.

60 His derivation of this proportion is hard to understand for us. For example, the editor of complete works of Daniel Bernoulli writes as follows. "What Bernoulli writes is difficult to follow in detail" (Daniel Bernoulli, *Werke 3*, Introduction, p.43.)


62 "Au reste j’ai été charmé de votre Théorème de la conservation des moments rotatoires, mais ce qui m’a poqué c’est que si j’avais fait un peu de reflexion sur mes equations, je l’aurais trouvé aussi. Car dans toutes les questions de même nature que j’ai donné à l'Académie, je suis arrivé deux equations finales dont l’une est celle qu'on peut trouver par la conservation des forces vives et dont l’autre est la même que celle que donne votre conservation des moments rotoartes." (Euler, *Opera Omnia*, ser.4A, vol.6, *Correspondance de Leonhard Euler avec A. C. Clairaut, J. d’Alembert et J. L. Lagrange*, Clairaut à Euler (No. 19), p.146.)

He derives an equation of a rotating body as below. He puts the rotating velocity of a point in the body ou, whose distance from the axis is f. He supposes that the point increases its velocity du during an infinitesimal rotation do. A point A has a velocity OA·ou/f and this velocity is produced by free fall from the height OA²·w/ff. During an infinitesimal rotation do, the point A proceeds OA·do, and gains an infinitesimal increase in height OA²·du/ff. Therefore, the particle is acted by a force along the direction Aa,

\[
\text{OA}^2 \cdot \text{du}/\text{ff} : \text{OA} . \text{d}/\text{O} = \text{A} . \text{OA} . \text{du}/\text{ffdo}
\]

Similarly, points B and C receive forces B·OB·du/ffdo and C·OC·du/ffdo, respectively. Therefore, the sum of moment of force with respect to the axis O is

\[
\text{du}/\text{ffdo} \cdot \text{(A} . \text{OA}^2 + \text{B} . \text{OB}^2 + \text{C} . \text{OC}^2 + \ldots\).
\]

By designating external moment of force as Pg, he gets the equation for the acceleration of a rotating body,

\[
\text{du} = \text{Pg} . \text{ffdo}/(\text{A} . \text{OA}^2 + \text{B} . \text{OB}^2 + \text{C} . \text{OC}^2 + \ldots).\]

By putting \(\text{A} . \text{OA}^2 + \text{B} . \text{OB}^2 + \text{C} . \text{OC}^2 + \ldots = \text{Mkk} \) (moment of inertia), he gives

\[
\text{du} = \text{Pg} . \text{ffdo}/\text{Mkk} \ldots(1).\]

By using the relation \(\text{du} = 2\text{ff} . \text{do} . \text{ddo}/\text{dt}^2\), which derives from \(\text{u} = \text{ffdo}/\text{dt}^2\), he obtains

\[
\text{Mkk} . \text{ddo}/\text{ffdo} = 2\text{Mkk} . \text{ddo}/\text{dt}^2.\]

Therefore, he rewrites (1) into another form

\[
2\text{Mkk} . \text{ddo}/\text{dt}_x = \text{Fx} \ldots(2).\]

64 Christiane Vilain, "La question du “Centre d’oscillation” de 1703 à 1743,” to be published in *La Revue Physik*.

65 Euler, ibid., §.54, p.126.

66 Euler, E86, p.75.


69 Daniel Bernoulli, *Werke 3*, General Introduction, p.XXVI.
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(1) motion=velocity of a mass(T.D.p.73).
(2) momentum du mouvement circulatoire.
(3) Left side...“omnium harum virtum summa momentorum ad axem”; Mkk...“momentum inertiae corporis”.
* D’Arcy also solves the same problem by using the conservation of angular momentum (the conservation of areal velocity) and the conservation of vis viva (1746).